Finitistic Dimensions of Monomial Algebras

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1 Introduction

The ideas of homology and homological algebra go back to the late 19^{th} century, when Poincaré [21] in 1895 introduced simplicial homology as a powerful invariant for certain classes of topological spaces. But a few years earlier, Hilbert [14] had already proven his so-called syzygy theorem:

Hilbert's Syzygy Theorem. If k is a field, then $\operatorname{gl\,dim} k[X_1, \ldots, X_n] = n$.

In his paper, Hilbert introduced the term "syzygy" in the context in which it is understood today; namely, given a module M over a ring R, a syzygy of M is any kernel of an epimorphism $P \twoheadrightarrow M$, where M is a projective R-module. Such a syzygy is unique up to a projective direct summand and "measures" the discrepancy of M from being projective.

Following the advent of category theory by Eilenberg and Mac Lane [8], progress in homological algebra increased dramatically. One very important result from the 1950's, connecting homological algebra with algebraic geometry, is the following:

Theorem. (Auslander-Buchsbaum-Serre, 1956; see [2] and [24].) Let V be an affine algebraic variety over an algebraically closed field with coordinate ring R. Then the global dimension of R is finite if and only if V is smooth. If this is the case, then $\operatorname{gl\,dim} R = \dim V$. (That the global dimension of R is bounded above by n means that n-fold iteration of the process of taking syzygies of R-modules always leads to a projective module.)

A consequence of this is the following result on localizations:

Corollary. (Auslander-Buchsbaum, 1959; see [4].) If V is a smooth affine algebraic variety over an algebraically closed field, then every localization of its coordinate ring is a unique factorization domain.

The general idea behind homological algebra is to associate certain invariants to objects in a category. (In our case, these categories will be categories of modules over finite-dimensional algebras.) In particular, a key object of study is a chain complex

$$\cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \cdots$$

of objects of some abelian category (frequently a category of modules over some ring), i.e. a collection of objects and maps as above so that $f_{n+1} \circ f_n = 0$ for all n. We then define the n^{th} homology

$$H_n = \frac{\ker(f_n)}{\operatorname{Im}(f_{n+1})}$$

of our chain complex. Typical examples of such homology groups are the Ext and Tor groups. The hope is that those objects which have trivial (or very straightforward) invariants can be easily understood; more complicated invariants measure the deviation from the easily understood objects. Far more information on homological algebra can be found, for example, in [25].

In these notes, our primary invariant of a module M will be the projective dimension of M, namely the least $n \in \mathbb{N}$ such that the n^{th} iterated syzygy is projective. The projective dimension is sometimes a good measure of the complexity of a module, provided that it is finite. However, as we shall see in the example on page 11, sometimes the projective dimension of a very readily understood module is infinite. Hence we frequently ignore modules of infinite projective dimension.

In response to this fact, the next step is to define the little and big left finitistic dimensions of a ring. The little (respectively, big) left finitistic dimension of a ring is the supremum of the projective dimensions of the finitely generated (respectively, arbitrary) left modules of *finite* projective dimension. One can immediately ask interesting questions about these finitistic dimensions: Do the little and big finitistic dimensions always agree? Are they always finite?

It turns out that both of these questions have been answered in the negative, even if we require the ring to be commutative and noetherian. The little and big finitistic dimensions of a commutative noetherian local ring agree if and only if the ring is Cohen-Macaulay (see [6]). This result follows from results of Auslander and Buchsbaum [3]. If R is a commutative noetherian ring, then the big finitistic dimension of R is equal to its Krull dimension (see [9]). Some examples of commutative noetherian rings with infinite Krull dimension were provided by Nagata [20].

However, the questions remained open in the case of noncommutative artinian rings for some time. Bass publicized them as problems in [5]. Shortly thereafter, they became known as the two finitistic dimension conjectures.

Finitistic Dimension Conjectures.

(I) If Λ is a finite-dimensional algebra, then the big and little finitistic dimensions of Λ coincide.

(II) If Λ is a finite-dimensional algebra, then the big finitistic dimension of Λ is finite.

At first there were some (very easily proved) partial positive results.

- Bass in [5] showed that if the little finitistic dimension is zero, then the big finitistic dimension is also zero (i.e. Conjecture I holds in this case).
- Mochizuki in [19] showed that if J is the Jacobson radical of Λ and $J^2 = 0$, then Conjecture II holds.

Little progress was then made until the early 1990's. Then many results began to appear.

- If Λ is a monomial algebra, then Conjecture II holds. This was shown first by Green, Kirkman, and Kuzmanovich in 1991 (see [13]).
- If $J^3 = 0$, then Conjecture II holds. This was shown by Green and Huisgen-Zimmermann in 1991 (see [12]).
- Conjecture I is false, even for monomial algebras. Huisgen-Zimmermann in 1992 found some counterexamples (see [16]). The first such examples are, in finite, finite dimensional monomial algebras, the class of algebras we are addressing in these notes.

2 Path Algebras Modulo Relations and Monomial Algebras

Definition. Let Q be a quiver (that is, a finite directed multigraph) and K a field. Define KQ to be the K-vector space having as basis all paths in Q including those of length 0. KQ carries a K-algebra structure, with multiplication defined as follows: if p and q are paths in KQ, set

$$pq = \begin{cases} p \text{ after } q & \text{if this is well-defined,} \\ 0 & \text{otherwise.} \end{cases}$$

Extend the multiplication bilinearly.

Therefore KQ is a K-algebra with identity given by $1 = \sum_{i=1}^{n} e_i$, where $e_1, \ldots, e_n$ are the paths of length 0, which we identify with the vertices themselves.

Examples and Remarks.

(1) Suppose that Q is the quiver

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

Then KQ is isomorphic, as a K-algebra, to the algebra of lower triangular $n \times n$ matrices.

(2) Suppose that Q is the quiver

$$\begin{tikzcd} \bullet \arrow[loop left] \arrow[loop right] \end{tikzcd}$$

Then KQ is isomorphic to the free algebra $K\langle x, y\rangle$ on two generators.

(3) Suppose that Q is the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend right] & 2 \end{tikzcd}$$

Then KQ is called the Kronecker algebra, and it is isomorphic to the matrix algebra

$$\begin{pmatrix} K & 0 \\ K \oplus K & K \end{pmatrix}.$$

(4) KQ is a finite dimensional algebra if and only if Q contains no oriented cycles.

Definition. An ideal $I \subseteq KQ$ is said to be **admissible** if it consists of linear combinations of paths of length at least 2, and there exists a positive integer L such that all paths of length L belong to I. (Hence all paths of length at least L belong to I.)

Note. KQ/I is a finite dimensional algebra whenever $I \subseteq KQ$ is an admissible ideal.

If I is an admissible ideal, we call KQ/I a path algebra modulo relations, or sometimes simply a path algebra.

Definition. We say that two rings R and S are **Morita equivalent** (and we write $R \approx S$) if there is an additive equivalence of categories between R-Mod and S-Mod. (If R and S are Morita equivalent, then there is also an additive equivalence of categories between Mod-R and Mod-S.)

Remark. If $R \approx S$, then the representation theories of R-Mod and S-Mod are interchangeable.

Theorem. (Gabriel) Suppose that K is algebraically closed. Then every finite dimensional K-algebra A is Morita equivalent to some path algebra modulo relations KQ/I. Moreover, Q is completely determined (up to quiver isomorphism) by A; I, however, is not unique in general.

For a proof, see §4.3 of [10].

Elementary Remarks. Let $\Lambda = KQ/I$, where $I \subseteq KQ$ is an admissible ideal. We identify any path of length zero with its residue class.

(1) If Q has exactly n vertices $e_1, \ldots, e_n$, then the $e_i \in \Lambda$ form a full sequence of primitive idempotents, i.e. all e_i are primitive, $e_i e_j = \delta_{ij} e_i$, and $1 = \sum_{i=1}^{n} e_i$. (Recall that an idempotent $e \neq 0$ is called primitive if whenever we write $e = e' + e''$ as the sum of two idempotents with $e'e'' = e''e' = 0$, then either $e' = 0$ or $e'' = 0$.)

(2) If $e_1, \ldots, e_n \in \Lambda$ are the full sequence of primitive idempotents of Λ, then $\Lambda e_1, \ldots, \Lambda e_n$ are precisely the indecomposable projective left Λ-modules up to isomorphism, and $\Lambda e_i \not\cong \Lambda e_j$ for $i \neq j$. Moreover, every projective in Λ-Mod is a direct sum of copies of the Λe_i's.

(3) Let J denote the Jacobson radical of Λ. Then $S_i = \Lambda e_i / J e_i = K(e_i + J e_i) \cong K e_i$ (this is a vector space isomorphism, as S_i is 1-dimensional) are all the simple modules up to isomorphism, and $S_i \not\cong S_j$ for $i \neq j$.

(4) J is a 2-sided ideal generated by the $\alpha + I$, where α is an arrow in Q. From now on, if p is a path in KQ, we call $p + I$ a path in Λ and often write p for $p + I$. Then

$$J = \left\{ \sum_{\text{finite}} k_i p_i \;\middle|\; k_i \in K, p_i \in \Lambda \text{ are paths of positive length} \right\}.$$

(Note that one cannot in general talk about the "length" of a path, but one can talk about paths of "positive length.") Suppose I contains all paths of length L. Then $J^L = 0$, so J is nilpotent. Moreover,

$$\Lambda / J = \bigoplus_{i=1}^{n} S_i.$$

Definition. Let M be in Λ-Mod, and let J be the Jacobson radical of Λ. Then P in Λ-Mod is called a **projective cover** of M if there is an epimorphism $f : P \twoheadrightarrow M$ with $\ker(f) \subseteq JP$.

Theorem. If Λ is any finite dimensional algebra, then every M in Λ-Mod has a projective cover. If M is finite dimensional, then so is P, and

$$\dim_K(P) = \min\{\dim Q \mid Q \text{ is a projective allowing for an epimorphism } Q \twoheadrightarrow M\}.$$

Furthermore, P and $\ker(f)$ are uniquely determined up to isomorphism by M.

For a proof, see Theorem 28.4 of [1].

Definition. If $f : P \to M$ is a projective cover, we call $\ker(f)$ the **first syzygy** of M and write $\ker(f) = \Omega^1(M)$.

Definition.

- A path algebra $\Lambda = KQ/I$ is called a **monomial algebra** if I can be generated by some (finite) set of paths.
- The paths of Q carry the following partial order: we say that $p \leq q$ if p is a right subpath of q (i.e. $q = p'p$ for some path p'). Since Λ is a monomial algebra, this partial order induces a partial order on the set of nonzero paths in Λ.

Examples.

(1) If $\Lambda = KQ/I$ is a path algebra modulo relations with $J^2 = 0$, then Λ is a monomial algebra. For example, if Q is the quiver

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

then

$$\frac{KQ}{\langle \text{all paths of length } 2 \rangle}$$

is a monomial algebra.

(2) If Q is the quiver

$$\begin{tikzcd}
 & 2 \arrow[dr, "\beta"] & \\
1 \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & 4 \\
 & 3 \arrow[ur, "\delta"'] &
\end{tikzcd}$$

then $\Lambda = KQ/\langle \beta\alpha - \delta\gamma \rangle$ is not a monomial algebra. In fact, it is not even isomorphic to a monomial algebra.

(3) Let Q be the quiver

$$\begin{tikzcd}
1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \arrow[r, "\gamma"] & 3
\end{tikzcd}$$

and $\Lambda = KQ/\langle \gamma\alpha - \gamma\beta \rangle$. Then Λ *is* isomorphic to a monomial algebra. Let $\tilde{I} = \langle \gamma\beta \rangle$, and set $\tilde{\Lambda} = KQ/\tilde{I}$. Then $\Lambda \cong \tilde{\Lambda}$ as K-algebras. An explicit isomorphism is given by sending $e_i + I \mapsto e_i + \tilde{I}$, $\alpha + I \mapsto \alpha + \tilde{I}$, $\beta + I \mapsto (\alpha+\beta) + \tilde{I}$, $\gamma + I \mapsto \gamma + \tilde{I}$ and extending linearly.

(4) If Q is the quiver

$$\begin{tikzcd}
\bullet \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"]
\end{tikzcd}$$

and $\Lambda = KQ/\langle \alpha^2, \beta^2, \alpha\beta - \beta\alpha \rangle$, then Λ is *not* isomorphic to a monomial algebra. This will follow from a theorem below.

Proposition. If $p_1, \ldots, p_r$ are nonzero paths in Λ such that, for $i \neq j$, p_i is not a right subpath of p_j, then

$$\sum_{i=1}^{r} \Lambda p_i = \bigoplus_{i=1}^{r} \Lambda p_i.$$

Proof. Suppose $\sum_{i=1}^{r} \lambda_i p_i = 0$. Since no p_j is a right subpath of p_1, we must have $\lambda_1 p_1 = 0$, for otherwise there would be no other summand to cancel out $\lambda_1 p_1$. Similarly, $\lambda_2 p_2$, and, in general, each $\lambda_i p_i = 0$. Hence $\sum_{i=1}^{r} \Lambda p_i$ is actually a direct sum, as desired.

Comment. "Path length" in Λ is in general not well-defined. Suppose Q is the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & 2 \arrow[rr, "\beta"] & & 5 \\ & 3 \arrow[r, "\delta"'] & 4 \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

and $I = \langle \beta\alpha - \varepsilon\delta\gamma \rangle$ and $\Lambda = KQ/I$. Then $\beta\alpha + I$ does not have a well-defined length, as $\beta\alpha + I = \varepsilon\delta\gamma + I$. However, if Λ is a monomial algebra, then path length *is* well-defined. Furthermore, if Λ is a monomial algebra, then the nonzero paths of Λ form a K-basis for Λ.

Proposition. Suppose Q is a quiver, and suppose I is an ideal of KQ generated by paths. Suppose m is the maximum length of any path in some generating set for I, and suppose there are h paths in KQ of length m. Then I is admissible if and only if for every path p of length $m + h$ in KQ, $p \in I$. (That is, p can be written as $p'qp''$, where p', q, and p'' are paths, and q is in the generating set for I.)

Proof. The reverse direction is clear. For the forward direction, suppose that an ideal I is admissible, but there is a path p of length at least $m+h+1$ that is not in I. Let p_i be the subpath of length m consisting of the i^{th} through $(m+i-1)^{\text{th}}$ arrows in p. (That is, $p = q_i p_i r_i$, where the length of q_i is $i-1$ and the length of p_i is m.) There are at least $h+1$ such subpaths p_i of p. By the pigeonhole, then, there exist i and j so that $p_i = p_j$. Assume without loss of generality that $i < j$, and let $p = q_i s_i p_j r_j$. Then all paths of the form $q_i s_i^k p_j r_j$ are not in I. Hence I is not admissible.

3 Basics on Homological Dimensions

Definition. Let M be a left Λ-module.

(1) A **projective resolution** of M is an exact sequence of left Λ-modules

$$\cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is projective.

(2) A **minimal projective resolution** of M is an exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr, "f_2"] \arrow[dr, "\bar{f}_2"'] & & P_1 \arrow[rr, "f_1"] \arrow[dr, "\bar{f}_1"'] & & P_0 \arrow[r, "f_0"] & M \to 0 \\
& & \ker(f_1) \arrow[ur, hook] & & \ker(f_0) \arrow[ur, hook] & &
\end{tikzcd}$$

where all P_i's are projective and each $f_i : P_i \to P_{i-1}$ gives rise to a projective cover $\bar{f}_i : P_i \to \mathrm{Im}(f_i) = \ker(f_{i-1})$ of $\ker(f_{i-1})$. Since projective covers are unique up to isomorphism, minimal projective resolutions are essentially unique.

(3) The **projective dimension** of M, written $\mathrm{p}\dim M$, is the length of a minimal projective resolution of M. Note that this length may be infinite, in which case we write $\mathrm{p}\dim M = \infty$.

(4) If

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

is a minimal projective resolution of M, then define the i^{th} **syzygy** of M to be $\Omega^i(M) = \ker(f_{i-1})$. Hence we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr, "f_2"] \arrow[dr, "\bar{f}_2"'] & & P_1 \arrow[rr, "f_1"] \arrow[dr, "\bar{f}_1"'] & & P_0 \arrow[r, "f_0"] & M \to 0. \\
& & \Omega^2(M) \arrow[ur, hook] & & \Omega^1(M) \arrow[ur, hook] & &
\end{tikzcd}$$

(5) The **left global dimension** of Λ is $\ell\ \mathrm{gl}\dim\Lambda = \sup\{\mathrm{p}\dim M \mid M \text{ is a left } \Lambda\text{-module}\}$. The **right global dimension** $\mathrm{r}\ \mathrm{gl}\dim\Lambda$ is defined analogously.

Remarks.

(1) Suppose $\mathrm{p}\dim M \geq d$. Then $\mathrm{p}\dim M = d + \mathrm{p}\dim \Omega^d(M)$.

(2) If $M = \bigoplus_{i\in I} M_i$, then $\mathrm{p}\dim M = \sup\{\mathrm{p}\dim M_i \mid i \in I\}$.

(3) If Λ is a finite-dimensional algebra, then $\ell\ \mathrm{gl}\dim\Lambda = \mathrm{r}\ \mathrm{gl}\dim\Lambda = \sup\{\mathrm{p}\dim S \mid S \text{ is a simple } \Lambda\text{-module}\}$. We simply write $\mathrm{gl}\dim\Lambda$ in the future.

Example. Suppose $\Lambda = KQ/I$, where Q is the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha_1"] & 2 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{n-1}"] & n \end{tikzcd}$$

and $I = \langle$all paths of length 2$\rangle$. The simple modules are $S_i = \Lambda e_i / J e_i$ for $i = 1, \ldots, n$. Since S_n is projective, p dim $S_n = 0$. A minimal projective resolution of S_{n-1} is

$$\begin{tikzcd} 0 \arrow[r] & \Lambda e_n \arrow[rr] \arrow[dr, two heads] & & \Lambda e_{n-1} \arrow[r, two heads, "f_0"] & S_{n-1} \arrow[r] & 0 \\ & & Je_{n-1} \cong S_n \arrow[ur, hook] & & & \end{tikzcd}$$

so p dim $S_{n-1} = 1$. A minimal projective resolution of S_{n-2} is

$$\begin{tikzcd} 0 \arrow[r] & \Lambda e_n \arrow[rr] \arrow[dr, two heads] & & \Lambda e_{n-1} \arrow[rr] \arrow[dr, two heads] & & \Lambda e_{n-2} \arrow[r] & S_{n-1} \arrow[r] & 0 \\ & & Je_{n-1} \cong S_n \arrow[ur, hook] & & Je_{n-2} \cong S_{n-1} \arrow[ur, hook] & & & \end{tikzcd}$$

so p dim $S_{n-2} = 2$. Inductively, p dim $S_i = n - i$, so gl dim $\Lambda = n - 1$.

It turns out that gl dim Λ is not always a good measure for the "complexity" of the category of Λ-modules.

Example. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd},$$

let $\Lambda = KQ/\langle \alpha^2 \rangle \cong K[X]/(X^2)$, and let J be the Jacobson radical of Λ. Then gl dim $\Lambda = \infty$, but the module theory of Λ is easy: Every left Λ-module is a direct sum of copies of $\Lambda e_1 =_\Lambda \Lambda$ and copies of $S = \Lambda e_1 / J e_1 \cong_\Lambda (\Lambda/J)$. Here p dim $\Lambda e = 0$ and p dim $S = \infty$. Since

$$\text{p dim} \left(\bigoplus_{i \in I} M_i \right) = \sup_{i \in I} \{ \text{p dim}\, M_i \},$$

every left Λ-module is either projective or else has infinite projective dimension.

Definition. Let Λ be a finite-dimensional algebra. Define the **left big** and **little finitistic dimensions** of Λ to be

$$\ell \text{ Fin dim}\,\Lambda := \sup\{\text{p dim}\, M \mid M \text{ in } \Lambda\text{-Mod},\ \text{p dim}\, M < \infty\},$$
$$\ell \text{ fin dim}\,\Lambda := \sup\{\text{p dim}\, M \mid M \text{ in } \Lambda\text{-mod},\ \text{p dim}\, M < \infty\},$$

respectively.

Remarks.

- In the above example, all finitistic left and right dimensions of Λ are zero.
- If gl dim $\Lambda < \infty$, then ℓ fin dim $\Lambda = \ell$ Fin dim $\Lambda =$ r fin dim $\Lambda =$ r Fin dim $\Lambda =$ gl dim Λ (because the global dimension is attained on a simple module).

Conjectures and related problems. (Bass 1960 popularized them first as problems, but they appear to have originated with Auslander and Buchsbaum.) Let Λ be a finite-dimensional algebra.

(1) ℓ fin dim $\Lambda < \infty$.

(2) ℓ fin dim $\Lambda = \ell$ Fin dim Λ.

(3) Problem: Specify sets $\mathscr{S}, \mathscr{S}'$ of modules in Λ-mod such that ℓ fin dim $\Lambda = \sup\{\text{p dim}\, M \mid M \in \mathscr{S}\}$ and ℓ Fin dim $\Lambda = \sup\{\text{p dim}\, M \mid M \in \mathscr{S}'\}$. Approximate ℓ fin dim Λ and ℓ Fin dim Λ, up to a specified error, from a *finite* set of data, given in terms of quiver and relations of Λ.

These notes address mainly this last problem for the case of monomial algebras.

Main Theorem. Suppose $\Lambda = KQ/I$ is a monomial algebra.

(1) For any M in Λ-Mod,
$$\Omega^2(M) \cong \bigoplus_{i \in I} \Lambda q_i,$$
where each q_i is a nonzero path in Λ of positive length, with repetitions permitted.

(2) ℓ Fin dim $\Lambda < \infty$. (Of course, this also implies that ℓ fin dim $\Lambda < \infty$.)

(3) If $\mathscr{S} = \{\Lambda q \mid q$ is a nonzero path of positive length in Λ with p dim $\Lambda q < \infty\}$, define

$$s = \begin{cases} -1 & \text{if } \mathscr{S} = \varnothing, \\ \max\{\text{p dim}\,\Lambda q \mid \Lambda q \in \mathscr{S}\} & \text{if } \mathscr{S} \neq \varnothing. \end{cases}$$

Then

$$s + 1 \le \ell \text{ fin dim}\,\Lambda \le \ell \text{ Fin dim}\,\Lambda \le s + 2.$$

Note that $\mathscr{S}$ is finite, since Λ contains only finitely many nonzero paths.

Part of Proof. We first show (1)⇒(2). Suppose M is in Λ-Mod with p dim $M < \infty$. Without loss of generality, we may assume p dim $M \ge 2$. Then

$$\Omega^2(M) \cong \bigoplus_{i \in I} \Lambda q_i,$$

where each q_i is a nonzero path in Λ of positive length, with repetitions permitted, for certain paths $q_i \neq 0$ in Λ with p dim $\Lambda q_i < \infty$. Since there are only finitely many nonzero paths of positive length in Λ, p dim $\Omega^2(M) = \sup\{\text{p dim}\,\Lambda q_i \mid i \in I\}$ is finite. Now p dim $M = 2 + \text{p dim}\,\Omega^2(M) \le 2 + s$, where s is as in (3).

We now show (1)⇒(3). We know that

$$s \le \ell \text{ fin dim}\,\Lambda \le \ell \text{ Fin dim}\,\Lambda \le s + 2.$$

We need only show now that we may safely replace s with $s+1$ on the left in the above inequality. For $\mathscr{S} \neq \varnothing$, $s + 1 = 0$, and the inequality is trivial. So let $\mathscr{S} \neq \varnothing$, and suppose q is a nonzero path of positive length in Λ with p dim $\Lambda q = s$. Let q be a path starting in e. Then $\Lambda q \subseteq Je$. Hence, if $X = \Lambda e/\Lambda q$, then

$$\Lambda e \xrightarrow{\text{can}} X = \frac{\Lambda e}{\Lambda q}$$

is the projective cover of X. [Note that $\ker(\text{can}) \subseteq Je$.] Thus $\Omega^1(X) = \ker(\text{can}) = \Lambda q$ has projective dimension s, so p dim $X = s + 1$ by Remark 3 on page 10.

For a proof of (1), see [15].

We now need some tools for dealing with examples. We will not provide proofs for these, but we will point out references where arguments can be found.

Proposition. If $\Lambda = KQ/I$ if a monomial algebra with Jacobson radical J, then for any Λ-module M, either $\text{p dim}\, M \leq \dim_K J + 1$ or $\text{p dim}\, M = \infty$. In other words, $\ell \text{ fin dim}\, \Lambda \leq \ell \text{ Fin dim}\, \Lambda \leq \dim_K J + 1$.

Proof. We first note that if $\Omega^k(M) = \bigoplus_{i \in I} M_i$, then

$$\text{p dim}\, M = k + \sup\{\text{p dim}\, M_i \mid i \in I\}. \qquad (*)$$

So suppose M has finite projective dimension, but that $\text{p dim}\, M \geq J + 2$. Consider the sequence $(\Omega^2(M), \Omega^3(M), \Omega^4(M), \ldots)$. By the Main Theorem, each $\Omega^k(M)$ above is a direct sum of modules generated by paths. There are only $\dim_K J$ different paths in Λ, so the supremum in $(*)$ is actually a maximum (i.e. the supremum is achieved). Let p_k be some path in Λ for which $\text{p dim}\, M = k + \text{p dim}\, \Lambda p_k$ (which exists for $k \geq 2$ by $(*)$ and the Main Theorem). Now consider the sequence $(p_2, p_3, \ldots, p_{J+2})$. By the pigeonhole principle, there must exist i and j (with $i < j$) so that $p_i = p_j$. But then Λp_i must be a direct summand of $\Omega^k(M)$ whenever M is of the form $i + r(j - i)$. Hence there are infinitely many k for which $\Omega^k(M)$ is nonzero. Therefore $\text{p dim}\, M = \infty$, contradicting our hypothesis.

Auxiliary facts.

(1) Krull-Schmidt Theorem, weak version (see [1], Theorem 12.9). Every finitely generated module is a finite direct sum of indecomposable modules. Furthermore, if M and N are finitely generated Λ-modules, with $M = M_1 \oplus \cdots \oplus M_r$ and $N = N_1 \oplus \cdots \oplus N_s$, where each M_i and N_j is indecomposable, then $M \cong N$ if and only if $r = s$ and there exists some $\pi \in S_r$ with $M_i \cong N_{\pi(i)}$ for all i.

(2) (See [5].) For a left Λ-module M, define the socle of M to be $\text{soc}(M_\Lambda) = \sum_S S$, where the sum is taken over all simple submodules of M. Then $\ell \text{ fin dim}\, \Lambda = 0$ if and only if $\text{soc}(\Lambda_\Lambda)$ contains an isomorphic copy of every simple *right* Λ-module.

(3) A **string algebra** (see [18]) is a monomial algebra $\Lambda = KQ/I$ such that

(i) For every vertex e of Q, there are at most two arrows entering e and at most two arrows leaving e.

(ii) If $\alpha : e \to e'$ is an arrow in Q, and β_1 and β_2 are arrows ending in e, then either $\alpha\beta_1 \in I$ or $\alpha\beta_2 \in I$; if γ_1 and γ_2 are two arrows starting in e', then either $\gamma_1\alpha \in I$ or $\gamma_2\alpha \in I$.

String algebras were originally studied by Gelfand and Ponomarev in [11], who showed that the indecomposable finitely generated modules over specific string algebras can be explicitly described. They have been studied further by Ringel in [22] and by Donovan and Freislich in [7].

(4) If Λ is a string algebra, then the finitely generated indecomposable modules can be described explicitly.

Example. Suppose $\Lambda = KQ/\langle \text{all paths of length } 2\rangle$, where Q is the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\alpha"] \arrow[r, "\beta"] & 2. \end{tikzcd}$$

Claim.

(a) ℓ fin dim $\Lambda = 1$.

(b) r fin dim $\Lambda = 0$.

We first show (b). We see that $\Lambda\alpha \cong S_1$ and $\Lambda\beta \cong S_2$ are both contained in ${}_\Lambda\Lambda$. So $\operatorname{soc}({}_\Lambda\Lambda) \cong S_1 \oplus S_2^2$. Therefore by (3) above, r fin dim $\Lambda = 0$.

For (a), we note that Λ is a string algebra. All the indecomposable finitely generated left Λ-modules are (up to isomorphism) S_1 (which has infinite projective dimension), S_2 (which has projective dimension zero), Λe_1 (which has projective dimension zero), $\Lambda e_1/\Lambda\alpha$ (which has $\Omega^1(\Lambda e_1/\Lambda\alpha) = \Lambda\alpha \cong S_1$ and hence infinite projective dimension), and $\Lambda e_1/\Lambda\beta$ (which has $\Omega^1(\Lambda e_1/\Lambda\beta) \cong S_2$ and hence projective dimension one); see e.g. [22] for a general description of the finitely generated indecomposable modules over a string algebra. By the first of the auxiliary facts listed above, every finitely generated module is a finite direct sum of indecomposables. Thus ℓ fin dim $\Lambda = 1$ by Remark (2) on page 10.

Example. Fix $n \in \mathbb{N}$. We'll give an example of $\Lambda = KQ/I$ such that all simple left Λ-modules have infinite projective dimension, but ℓ fin dim $\Lambda = n$. (Here, again, r fin dim $\Lambda = 0$.) Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha_1"] \arrow[loop below, "\omega_1"] & 2 \arrow[r, "\alpha_2"] \arrow[loop below, "\omega_2"] & \cdots \arrow[r] & n \arrow[r, "\alpha_n"] \arrow[loop below, "\omega_n"] & n+1 \arrow[loop below, "\omega_{n+1}"] \end{tikzcd}$$

and $I = \langle \omega_i^2, \alpha_{j+1}\alpha_j, \alpha_j\omega_j \mid 1 \le i \le n+1, 1 \le j \le n \rangle$. Note that Λ is a string algebra. Then $\Lambda\omega_i$ is a direct summand of $Je_i = \Omega^1(S_i)$. (Indeed, $Je_i = \Lambda\omega_i \oplus \Lambda\alpha_i$.)

(a) Since $S_i \cong \Lambda\omega_i$, we see that S_i is isomorphic to a direct summand of $\Omega^1(S_i)$, which shows that p dim $S_i = \infty$.

(b) On the other hand, one checks that p dim $\Lambda e_1/\Lambda\alpha_1 = n$. Indeed,

$$\Omega^1(\Lambda e_1/\Lambda\alpha_1) = \Lambda\alpha_1 \cong \Lambda e_2/\Lambda\alpha_2,$$

and, more generally, for $i \le n$ we have

$$\Omega^1(\Lambda e_i/\Lambda\alpha_i) = \Lambda\alpha_i \cong \begin{cases} \Lambda e_{i+1}/\Lambda\alpha_{i+1} & \text{for } i < n, \\ \Lambda e_{n+1} & \text{for } i = n. \end{cases}$$

Thus

$$\Omega^n(\Lambda e_1/\Lambda\alpha_1) = \Lambda e_{n+1};$$

that is, the length of a minimal projective resolution of $\Lambda e_1/\Lambda\alpha_1$ is n. Hence ℓ fin dim $\Lambda \ge n$.

We will return to this example for a more intuitive graphical proof of the inequality "ℓ fin dim $\Lambda \ge n$."

We now return to an old example, namely Example 2 on page 8. We claimed that Λ is *not* a monomial algebra. Let $M = \Lambda e_1/\Lambda\beta\alpha$. Then $\Omega^1(M) \cong \Lambda\beta\alpha \cong S_1$, and

$$\Omega^2(M) \cong \Omega^1(S_1) = \frac{\Lambda x \oplus \Lambda y}{\Lambda\alpha x + \Lambda\beta y + \Lambda(\beta x - \alpha y)},$$

where $x = y = e_1$. This module is indecomposable but not cyclic. Then Theorem 3 on page 12 for second syzygies over monomial algebras proves the claim.

4 Tree Modules over Monomial Algebras, their Graphs, and their Syzygies

The algorithm which we will present in this section will allow us to compute the number s occurring in the Main Theorem of §2 from the quiver and relations of a monomial algebra Λ.

In fact, there is a computer program that can compute this number s once the user enters a quiver and an admissible ideal; see [23].

Definitions. Let $\Lambda = KQ/I$ be a monomial algebra.

(1) A **tree module** in Λ-mod rooted in a vertex e of Q is a module of the form

$$T \cong \frac{\Lambda e}{\sum_{v \in \mathcal{V}} \Lambda v} \ ,$$

where $\mathcal{V}$ is a set of paths in Λ which start at e. Note that $T = 0$ iff $e \in \mathcal{V}$. Moreover, the only simple tree module rooted at e is $S = \Lambda e/Je$ up to isomorphism.

(2) Given a tree module $T = \Lambda e/V$, where $V = \sum_{v \in \mathcal{V}} \Lambda v$, the **branches** of T are those paths in $\Lambda e \setminus V$ (i.e. those nonzero paths in Λ which start in e and do not belong to V) which are maximal under the partial order $\leq$ introduced on page 7.

Remarks. Let $T = \Lambda e / \sum_{v \in \mathcal{V}} \Lambda v$ be a nonzero tree module, where $\mathcal{V}$ is a set of nonzero paths in Λ which start in e. (Recall that we identify a full set $e_1, \dots, e_n$ of primitive idempotents of Λ with the vertices of Q, and suppose $e = e_t$.)

- Then $T/JT \cong S_t = \Lambda e_t/Je_t$ has dimension 1 as a K-vector space, and for $1 \le \ell \le L$,
$$\frac{J^\ell T}{J^{\ell+1}T} \cong \bigoplus_{i=1}^n S_i^{\tau_i^{(\ell)}},$$
where the $\tau_i^{(\ell)}$ are nonnegative integers with $\sum_{i=1}^n \tau_i^{(\ell)} = \dim_K J^\ell T/J^{\ell+1}T$.

- Suppose that $b_1, \dots, b_r$ are the branches of T. Then, given $\ell \in \{0, \dots, L\}$ and $i \in \{1, \dots, n\}$, the number of distinct right subpaths of length ℓ of the b_k which end in i is equal to $\tau_i^{(\ell)}$.

Definitions (continued).

(3) Suppose T is a nonzero tree module, say $T = \Lambda e/V = \Lambda e/\bigoplus_{v\in\mathcal{V}} \Lambda v$ as above. The **layered and labeled graph** $\mathfrak{g}(T)$ of T is the undirected rooted tree graph defined as follows:

(a) The graph $\mathfrak{g}(T)$ has $\dim_K T$ vertices, arranged in $L+1$ rows (numbered 0 to L from top to bottom) in the following manner:

- If $J^\ell T/J^{\ell+1}T \cong \bigoplus_{i=1}^n S_i^{\tau_i^{(\ell)}}$, the ℓ^{th} row of the graph contains precisely $\tau_i^{(\ell)}$ vertices labeled i.
- The graph $\mathfrak{g}(T)$ has edges only between vertices of adjacent rows and no multiple edges. Each edge from a vertex i to a vertex j is labeled by the name of an arrow $\alpha : i \to j$.

(b) In view of the preceding remarks, there is precisely one labeled undirected tree graph (up to graph isomorphism) with the specified vertex set such that all branches of T occur as edge paths when read from the top down. (Here we identify an edge path

$$e_t \overset{\alpha_1}{\text{———}} e_{i_1} \overset{\alpha_2}{\text{———}} e_{i_2} \text{———} \cdots \text{———} e_{i_{n-1}} \overset{\alpha_m}{\text{———}} e_{i_m}$$

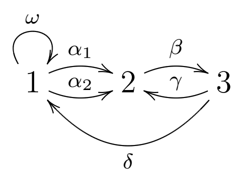
in $\mathfrak{g}(T)$ with the path $\alpha_m\alpha_{m-1}\cdots\alpha_1$. More specifically, if $JT/J^2T \cong \bigoplus_{i=1}^n S_i^{\tau_i^{(1)}}$, there are, for each i, precisely $\tau_i^{(1)}$ distinct arrows $e_t \to e_i$. These arrows give rise to one edge each from the vertex labeled t in row 0 to the $\tau_i^{(1)}$ vertices labeled i in row 1, and so forth.

Remark. The module $\Lambda e/Je$ is a tree module because

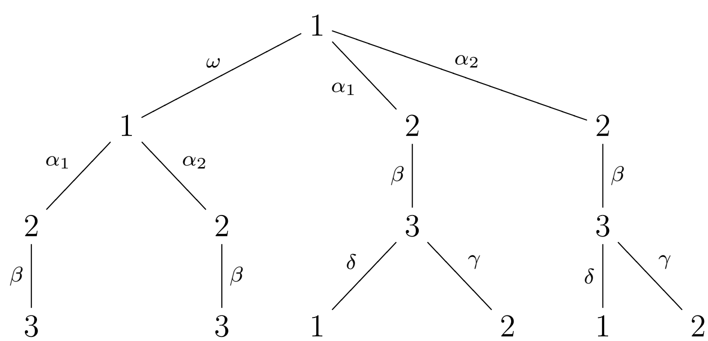
$$Je = \bigoplus \Lambda v,$$

where the direct sum is taken over all paths v of positive length starting at e. Moreover, if $T \neq 0$ is a tree module not isomorphic to $\Lambda e/Je$, then there is a nonzero submodule $U \subseteq T$ with $T/U \cong \Lambda e/Je$.

Example. Let Q be the quiver



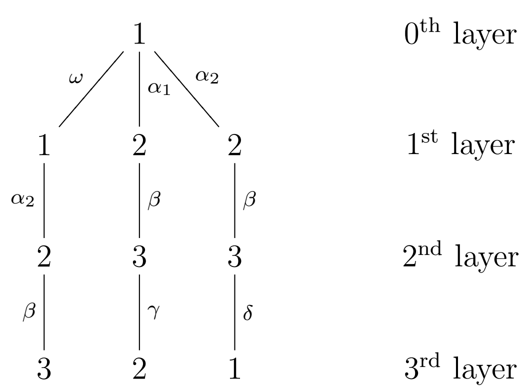
and $\Lambda = KQ/\langle \omega^2, \text{all paths of length } 4\rangle$. The graph of Λe_1 is



Now let

$$T = \frac{\Lambda e_1}{\Lambda\alpha_1\omega + \Lambda\delta\beta\alpha_1 + \Lambda\gamma\beta\alpha_2}.$$

The branches of T are $\beta\alpha_2\omega$, $\gamma\beta\alpha_1$, and $\delta\beta\alpha_2$. Thus $\mathfrak{g}(T)$ is



Remark. Every tree module is indecomposable. To see this, note that if T is a tree module, its unique maximal submodule is JT. Now suppose we had a decomposition $T = T_1 \oplus T_2$, where T_1 and T_2 are nonzero. Let U_1 and U_2 be maximal submodules of T_1 and T_2, respectively. Then $U_1 \oplus T_2$ and $T_1 \oplus U_2$ would be distinct maximal submodules of T. Hence T must be indecomposable.

Comments.

(1) If q is a nonzero path in Λ which ends in e, then Λq is a tree module with root e.

(2) Suppose $M = \bigoplus_{i=1}^{s} T_i$ is a direct sum of tree modules T_i. By the Krull-Schmidt Theorem (comment 3 on page 14), every decomposition of M into indecomposable modules is of the form $M_1 \oplus \cdots \oplus M_s$, with $M_i \cong T_i$.

So defining the layered and labeled graph $\mathfrak{g}(M)$ of M as the juxtaposition of the $\mathfrak{g}(T_i)$ is unambiguous.

Theorem. Let Λ be a monomial algebra.

(1) For every nonzero path q in Λ, the left ideal Λq is a tree module.

(2) For any Λ-module M, the syzygies $\Omega^k(M)$ for $k \geq 2$ are direct sums of tree modules.

(3) If M is a direct sum of tree modules, then $\Omega^1(M)$ is again a direct sum of tree modules.

Remarks.

(1) If $M = \Lambda e/V$ is a tree module, the branches of M are uniquely determined by the isomorphism class of M.

(2) If we know that M is a tree module, the branches of M determine M up to isomorphism.

(3) Up to isomorphism, the simple module $\Lambda e/Je$ is the unique tree module of minimal K-dimension with root e. In this case, e is the only branch of the tree module.

(4) Suppose Λ is a monomial algebra. Then up to isomorphism, Λe is the unique tree module of maximal K-dimension with root e.

(5) If M is a non-simple tree module, then all branches of M have positive length.

From now on we assume that $\Lambda = KQ/I$ is a monomial algebra.

Proof of the Theorem.

(1) Let e be the endpoint of q. Consider the map $f : \Lambda e \to \Lambda q$ given by $\lambda e \mapsto \lambda eq = \lambda q$. Clearly f is an epimorphism, so $\Lambda q \cong \Lambda e/\ker(f)$. We will show that

$$\ker(f) = \sum_{v \in \mathcal{V}} \Lambda v,$$

where $\mathcal{V}$ consists of paths v in Λe with $vq = 0$ in Λ. Let $0 \neq \lambda e \in \ker(f)$; say $\lambda e = \sum k_i p_i$, where $k_i \in K^{\times}$, and the p_i are nonzero paths in Λe. Then $0 = f(\lambda e) = \sum k_i p_i q$. Suppose that $p_1 q, \ldots, p_r q \in \Lambda$ are the nonzero paths among the $p_i q \in \Lambda$. Since the $p_i q$ are linearly independent in Λ, the equality

$$\sum_{i=1}^{r} k_i p_i = 0$$

implies that $r = 0$. Thus $\lambda e = \sum k_i p_i$ as before, with $p_i q = 0$ for all i.

(2) This follows from part (1) and the structure theory for syzygies $\Omega^k(M)$, $k \geq 2$.

(3) It clearly suffices to prove the claim for a tree module M. Say $M = \Lambda e/V$, where $V = \sum_{v \in \mathcal{V}} \Lambda v$, and $\mathcal{V}$ is a set of nonzero paths of positive length in Λe. We know that the canonical epimorphism $\pi : \Lambda e \to M$ is a projective cover of M. Hence $\Omega^1(M) = \ker \pi = V$. Thus once we have shown that there exist $v_1, \ldots, v_r \in \mathcal{V}$ such that $V = \bigoplus_{i=1}^{r} \Lambda v_i$, we obtain $\Omega^1(M) = \bigoplus_{i=1}^{r} \Lambda v_i$, and each Λv_i is a tree module by part (1). To prove this, let $v_1, \ldots, v_r \in \mathcal{V}$ be the distinct minimal paths in $\mathcal{V}$ under $\leq$ (the partial order in part (2) of the Definition on page 7) because every path in $\mathcal{V}$ contains one of $\{v_1, \ldots, v_r\}$ as a

right subpath. Then $\sum_{i=1}^{r} \Lambda v_i = V$. Since $v_i \not\leq v_j$ whenever $i \neq j$, we conclude that $\sum_{i=1}^{r} \Lambda v_i = \bigoplus_{i=1}^{r} \Lambda v_i$.

For an important class of monomial algebras, the string algebras, defined on page 14, we can improve on the theorem bounding the finisitic dimension of a monomial algebra from above and below in terms of the number s.

Theorem. If Λ is a finite-dimensional string algebra, then the number $s+1$ defined in the beginning of §3 is equal to ℓ fin dim Λ.

See [18] (especially Proposition 2 and Theorem 3) for a proof.

We now return to the example on page 16. The indecomposable projective left Λ-modules have the following graphs:

$$\Lambda e_1 \qquad \Lambda e_2 \qquad \cdots \qquad \Lambda e_n \qquad \Lambda e_{n+1}$$

$$\begin{tikzcd}
1 \arrow[d, dash, "\omega_1"'] \arrow[dr, dash, "\alpha_1"] & \\
1 & 2 \arrow[d, dash, "\omega_2"] \\
& 2
\end{tikzcd}
\quad
\begin{tikzcd}
2 \arrow[d, dash, "\omega_2"'] \arrow[dr, dash, "\alpha_2"] & \\
2 & 3 \arrow[d, dash, "\omega_3"] \\
& 3
\end{tikzcd}
\qquad
\begin{tikzcd}
n \arrow[d, dash, "\omega_n"'] \arrow[dr, dash, "\alpha_n"] & \\
n & n+1 \arrow[d, dash, "\omega_{n+1}"] \\
& n+1
\end{tikzcd}
\quad
\begin{tikzcd}
n+1 \arrow[d, dash, "\omega_{n+1}"'] \\
n+1
\end{tikzcd}$$

We see that p dim $S_i = \infty$ for $i = 1, 2, \ldots, n+1$. Thus $\mathscr{S} = \{\alpha_1, \ldots, \alpha_n\}$, and p dim $\Lambda\alpha_i = n - i$. Hence ℓ gl dim $\Lambda \geq n$.

We have mostly only discussed computing the finitistic dimensions to within an error of one. However, sometimes we can compute the finitistic dimensions exactly, both in cases in which the finitistic dimensions are equal to $s+1$ and those in which the finitistic dimensions are equal to $s+2$.

Theorem. Let s be as above, with $s \neq -1$, and let $q_1, \ldots, q_t$ be all the paths of positive length with $\operatorname{p\,dim} q_i = s$. Decompose each q_i as follows: let $q_i = \alpha_i p_i$, where α_i has length 1 and p_i is some path with length at least 0.

(1) If for all $j = 1, \ldots, t$ and all $w \in \operatorname{r.ann}_J q_j$, then $\ell \operatorname{fin\,dim} \Lambda = \ell \operatorname{Fin\,dim} \Lambda = s + 1$.

(2) If there is some j with $1 \leq j \leq t$ and a set of paths $A \subseteq \operatorname{r.ann}_J q_j$ with $p_j A \neq 0$ in Λ satisfying $\operatorname{p\,dim} \ell.\operatorname{ann}_\Lambda A < \infty$, then $\ell \operatorname{fin\,dim} \Lambda = \ell \operatorname{Fin\,dim} \Lambda = s + 2$.

For a proof, see [15].

References

[1] F. W. Anderson and K. R. Fuller. *Rings and categories of modules*, Second Edition. Graduate Texts in Mathematics **13**, Springer-Verlag, New York, New York, USA, 1998.

[2] M. Auslander and D. Buchsbaum. “Homological dimension in noetherian rings.” *Proc. Nat. Acad. Sci. USA* **42**, 36–38 (1956).

[3] M. Auslander and D. Buchsbaum. “Homological dimension in regular local rings.” *Trans. Amer. Math. Soc.* **85**, 390–405 (1957).

[4] M. Auslander and D. Buchsbaum. “Unique factorization in regular local rings.” *Proc. Nat. Acad. Sci. USA* **45**, 733–734 (1959).

[5] H. Bass. “Finitistic dimensions and a homological generalization of semi-primary rings.” *Trans. Amer. Math. Soc.* **95**, 466–488 (1960).

[6] W. Bruns and J. Herzog. *Cohen-Macaulay rings.* Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, England, 1993.

[7] P. W. Donovan and M.-R. Freislich. “The indecomposable modular representations of certain groups with dihedral Sylow subgroup.” *Math. Ann.* **238**, 207–216 (1978).

[8] S. Eilenberg and S. Mac Lane. “General theory of natural equivalences.” *Trans. Amer. Math. Soc.* **58**, 231–294 (1945).

[9] D. Eisenbud. *Commutative algebra with a view toward algebraic geometry.* Graduate Texts in Mathematics **150**, Springer-Verlag, New York, New York, USA, 2004.

[10] P. Gabriel. "Auslander-Reiten sequences and representation-finite algebras" in *Representation theory I (Proc. Ottawa, Carleton University, 1979)*, V. Dlab and P. Gabriel, eds, Lecture Notes in Mathematics **831**, Springer-Verlag, Berlin, Germany, 1980.

[11] I. M. Gelfand and V. A. Ponomarev. "Indecomposable representations of the Lorentz group." *Usp. Math. Nauk* **23**, 3–60 (1968).

[12] E. L. Green and B. Huisgen-Zimmermann. "Finitistic dimensions of artinian rings with vanishing radical cube." *Math. Z.* **206**, 505–526 (1991).

[13] E. L. Green, E. E. Kirkman, and J. J. Kuzmanovich. "Finitistic dimension of finite dimensional monomial algebras." *J. Algebra* **136**, 37–51 (1991).

[14] D. Hilbert. "Über die Theorie der algebraischen Formen." *Math. Ann.* **36**, 473–534 (1890).

[15] B. Huisgen-Zimmermann. "Predicting syzygies over monomial relations algebras." *Maunscripta Math.* **70**, 157–182 (1991).

[16] B. Huisgen-Zimmermann. "Homological domino effects and the first Finitistic Dimension Conjecture." *Invent. Math.* **108**, 369–383 (1992)

[17] B. Huisgen-Zimmermann. "A convenient source of homological examples over artinian rings" in *Ring theory (Proc. Bicentennial Ohio State–Denison Conf. 1992)*, S. K. Jain and S. T. Rizvi, eds. 364–372, World Scientific, Singapore, 1993.

[18] B. Huisgen-Zimmermann and S. O. Smalø. "The homology of string algebras I." *J. reine angew. Math.* **580**, 1–37 (2005).

[19] H. Mochizuki. "Finitistic global dimension for rings." *Pacific J. Math.* **15**, 249–258 (1965).

[20] M. Nagata. *Local rings.* Wiley, New York, New York, USA, 1962.

[21] H. Poincaré. "Analysis situs." *Journal de l'École Polytechnique, Ser. 2* **1**, 1–123 (1895).

[22] C. M. Ringel. "The indecomposable representations of the dihedral 2-groups." *Math. Ann.* **214**, 19–34 (1975).

[23] S. Ryerson. Syzygy (computer software), 2007.

[24] J.-P. Serre. "Sur la dimension homologique des anneaux et des modules Noethériens" in *Proc. Internat. Symposium on Algebraic Number Theory, Tokyo and Nikko 1955*, 175–189, Science Council of Japan, Tokyo, Japan, 1956.

[25] C. A. Weibel. *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, Cambridge, England, 1994.