Equilibrium Points of Potential Fields Produced by Positive Point Charges

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ABSTRACT

In his 1873 *Treatise on Electricity and Magnetism*, J.C. Maxwell gave without proof an upper limit on the number of equilibrium points on a potential field produced by positive point charges in 3-space. This paper will first present Maxwell's description of his claim, as well as the progress already made on proving it. We will then study Maxwell's claims piece by piece, proving certain aspects of Maxwells conjecture and producing the proof of the conjecture itself in simple cases. We will conclude by examining the nature of equilibrium points under more generalized conditions and provide insight on possible ways to further the progress made by this paper.

Contents

1 Introduction

Before presenting Maxwell's conjecture, let us first define some concepts related to the conjecture.

Definition 1.1. *Given a configuration of k point charges in 3-space, respectively with charges* $\{q_1, q_2, ..., q_k\}$ *and at locations* $\{p_1, p_2, ..., p_k\} \in R^3$ *, the* ***potential*** *of a given point* $p \in R^3$ *is defined as*

$$V(p) = \sum_{i=1}^{k} \frac{q_i}{\mid p - p_i \mid}$$

Definition 1.2. *Given a point* $p \in R^3$*, we call* p *an* ***equilibrium point*** *if* $\bigtriangledown V(p) = 0$*.*

We call $l \subset R^3$*, where* l *is connected and contains more than one point, an* ***equilibrium line*** *if* $\forall p \in l, \bigtriangledown V(p) = 0$*.*

A subset of R that is either an equilibrium point or an equilibrium line is called a ***body of equilibrium****.*

1.1 Types of Bodies of Equilibrium

In his Treatise, Maxwell described two type of equilibrium points.

Maxwell's descriptions of equilibrium points were generalized for all types of charged bodies of all types of charges, since this paper studies only cases where all charged bodies are positive point charges, the descriptions we give will be simplified.

For the purpose of more smoothly explaining Maxwell's ideas, some of the language used here were not used by Maxwell himself. Likewise, some language used by Maxwell will not be used here.

For more generalized descriptions of equilibrium points, see [4].

Definition 1.3. *For a configuration of point charges in 3-space, consider a value* $c \in R$*. We call* $A \subseteq R^3$ *a* ***positive region*** *(resp.* ***negative region****) in regards to* c *if* A *is connected,* $V(a) \geq c$ *(resp.* $V(a) < c$*)* $\forall a \in A$*, and* A *is not the subset of another set with the previous two properties.*

For a configuration of positive point charges, we see that if c were a sufficiently large number, the positive regions in regards to c would correspond to the locations of the charges, while the rest of R^3 would consititute one negative region.

Now allow c to decrease. As this happens, we will notice that the existing positive regions expand and new ones may form. As these regions expand, they may meet each other at a point or a line and go on to meld into a single positive region. Once c reaches a sufficiently low value, all of R^3 becomes a single positive region.

We will call bodies of equilibrium formed by two or more positive regions colliding **type I bodies of equilibrium**. According to Maxwell, given a configuration of positive k point charges, the number of type I bodies of equilibrium is thus bounded above by $k - 1$.

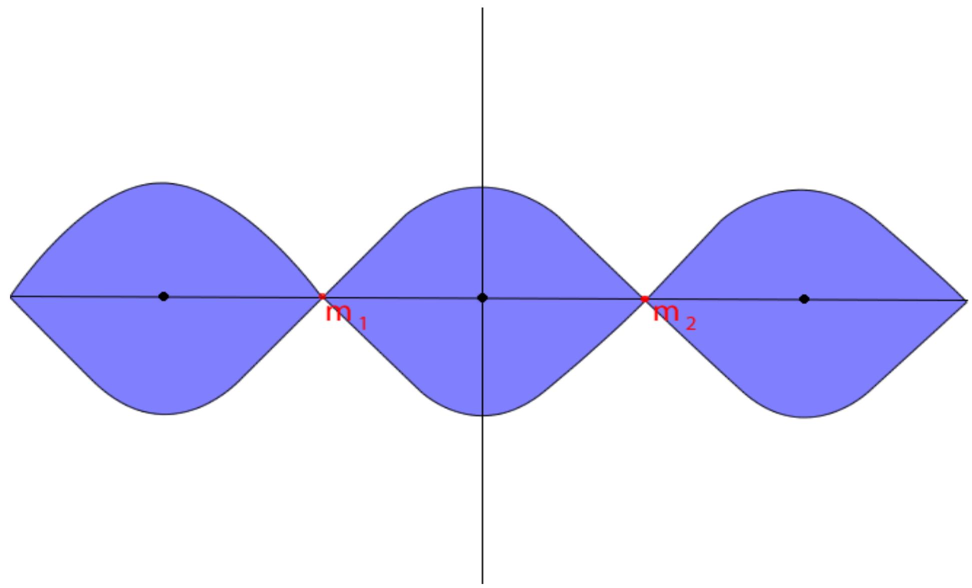

Figure 1
Examples of Type I Bodies of Equilibrium (at m_1 and m_2).
Positive regions coloured in blue. Black dots are point charges.

Bodies of equilibrium are not necessarily type I. Sometimes, as c decreases, a positive region may wrap around and meet itself in a ring shape, forming a body of equilibrium. The interior negative region within the positive ring may then shrink as c further decreases to form another equilibrium point.

We call bodies of equilibrium formed by positive regions colliding with themselves **type II bodies of equilibrium**.

According to Maxwell, given a configuration of k positive point charges, the number of type II bodies of equilibrium formed is bounded above by $(k-1)(k-2)$.

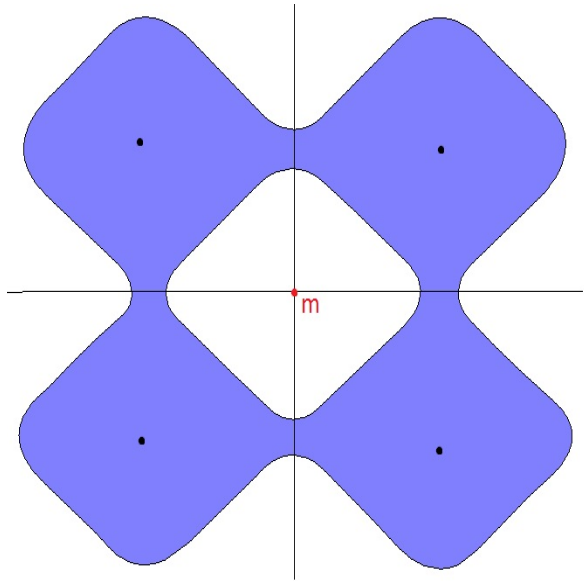

Figure 2
Example of Type II Body of Equilibrium (at m).
Positive regions coloured in blue. Black dots are point charges.

1.2 Special Note

There is a special note to be made about multiple equilibrium points appearing for the same $c \in R$. When such a case occurs, it is important to consider the equilibrium points in order rather than separately, for they may affect whether each body of equilibrium is type I or type II.

That is to say, for bodies of equilibrium $\{x_1, x_2, ...x_m\}$ produced at $c \in R$, for purposes of determining type of body of equilibrium, assume each x_i is produced for different values of c: x_1 occurs immediately before x_2, x_2 immediately before x_3, and so on. The typing of any single body may change depending on how the bodies are indexed, but the number of each type of body will remain the same.

To illustrate a situation where this matters, consider three point charges, each of value +1, located at (1,0,0), (-1,0,0) and (0,$\sqrt{3}$,0). When $c \approx 2.615$, we have three equilibrium points forming as shown in the figure below.

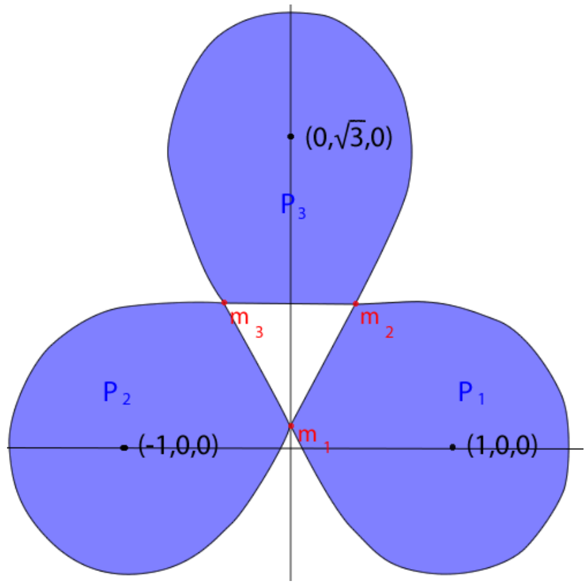

Figure 3
Cross section of $V(p)$ when z=0.
Positive regions coloured and labeled in blue.

If we examined equilibrium points m_1, m_2, m_3 separately, we would conclude that all three points are type I, with m_1 formed from positive regions P_1 and P_2 colliding, m_2 from P_1 and P_3, and m_3 from P_2 and P_3. However this is not the case.

We must instead observe these three points as though m_1, m_2, m_3 occur one after another. In doing this, we will have that m_1 is a type I body of equilibrium, formed from P_1, P_2 colliding; m_2 would also be type I, formed from $P_1 \cup m_1 \cup P_2$ and P_3 colliding. However, m_3 would, through this algorithm, be formed by $P_1 \cup m_1 \cup P_2 \cup m_2 \cup P_3$ colliding with itself, making it a type II point of equilibrium.

1.3 The Conjecture and Assumptions

Summing up the number of bodies of equilibrium from Section 1.1, we arrive at the following conjecture.

Conjecture 1.1. *Given a configuration of k positive point charges, there exist at most* $(k-1)^2$ *bodies of equilibrium.*

This is the conjecture bounding the number of bodies of equilibrium generally associated with Maxwell.

Those who have attempted to prove this conjecture do so after making two assumptions. The first assumption is that there are only finitely-many equilibrium points, thus making Maxwell's conjecture a bound on points of equilibrium rather than general bodies.

The second assumption is that all equilibrium points are non-degenerate, meaning they are all either local minima, local maxima, or a saddle points. We shall see later on that the possible topologies of equilibrium points can be reduced even further.

There are no proofs that either of these assumptions are true for general configurations of positive point charges. Nevertheless, even allowing for the existence of infinitely-many equilibrium points and degenerate equilibrium points, there has yet to be a counter-example of Maxwell's conjecture.

This paper will be making the same two assumptions on equilibrium points.

1.4 Progress So Far

Little progress has been made in proving this conjecture.

Through use of Khovanskii's Theorem, [2] has been able to find a bound of $4^{k^2}(3k)^{2k}$ for a general configuration of k positive point charges. Although this bound is effective, it is likely also an overestimate, producing a bound of about 1.4×10^{11} in the case of k=3.

Through usage of the principles behind Khovanskii's Theorem rather than the theorem itself, [2] was then able to produce a sharper bound of 12 in the case of k=3.

Through use of Bezout's Theorem, [3] has been able to find a bound of $(2^{k-1}(3k-2))^2$ for a general configuration of k positive point charges, but only when those k charges and their equilibrium points are arranged on a line or a plane.

As with [2], one will see that the bound produced by [3] is likely an overestimate, producing a bound of 784 in the case of k=3.

1.5 Contents of This Paper

While the final goal of both [2] and [3] was to work towards a proof for Conjecture 1.1, neither Khovanskii's Theorem or Bezout's Theorem closely relate to the train of thought Maxwell himself used to arrive at his conjecture. Both papers were attempts to examine Conjecture 1.1 through entirely different channels from Maxwell.

In this paper, however, we will break apart Maxwell's own approach piece by piece, examining each of the (also unproven) claims Maxwell made in the lead-up to Conjecture 1.1. The goal is to make progress towards proving the conjecture in a way Maxwell may have intended.

2 Type I Equilibrium Points

With the concept of positive regions and the types of equilibrium points fresh in our minds, let's jump immediately into the study of type I equilibrium points. To do so, we shall refer to one of the most well-known theorems in PDE's. For a proof of this theorem, see [6].

Theorem 2.1 (Maximum Principle)**.** *Given any open set $D \subset R^3$, let u be a function that is harmonic in D, continuous on $\bar{D}$, and non-constant. The maximum and minimum values of u on $\bar{D}$ are obtained on ∂D and nowhere else.*

Proposition 2.1. *On any set that has a non-empty interior and does not contain any point charges, the potential is harmonic, continuous, and non-constant.*

Proposition 2.1 tells us that we can apply the Maximum Principle to any open set not containing a charge. We shall introduce one more concept before using the Maximum Principle to help put on bound on the number of type I equilibrium points.

Definition 2.1. *A metric space (X, d) is **normal** if, for any two disjoint close sets $A, B \subset X$, there exist open sets $P, Q \subset X$ such that $A \subseteq P$, $B \subseteq Q$, and P, Q disjoint.*

Theorem 2.2. *R^n, with topology defined by the Euclidean metric, is normal.*

For a proof of Theorem 2.2, see [8]. It is now worth noting that by Definition 1.3, any positive region with respect to $c \in R$ is disjoint from any other positive region with respect to c. If they were not disjoint, then their union would be a positive region, making each of them not satisfy the last condition of Definition 1.3.

Proposition 2.2. *Given a configuration of positive point charges in R^3 and $c \in R$, any positive region with respect to c must contain at least one point charge.*

Proof. Since this proof is trivial for $c \leq 0$ (when we'd have a single positive region covering all of R^3), we'll only consider cases when $c > 0$ in this proof.

Let D be a positive region with respect to c not containing a charge. By definition 1.3, D is closed and disjoint from other positive regions.

Let $d \in D$ be a point at which the Potential obtains its maximum value in D. We know d exists since positive regions are bounded for $c > 0$. Since R^3 is normal, $\exists$ open set D^+ containing D and disjoint from other positive regions. Clearly, $d \in D^+$.

Since D^+ is open, $\exists$ ϵ such that $B_\epsilon(d) \subset D^+$. Since D^+ is disjoint from all positive regions besides D, for every $p \in B_\epsilon(d)$, either $p \in D$ or $V(p) < c$ (p is in a negative region).

Now note that $d \in D \Rightarrow V(d) \geq c$, and also $V(d) \geq V(p)$ $\forall$ $p \in D$ by our choice of d. Therefore, the potential obtains its maximum in $B_\epsilon(d)$ at d, which is clearly not on its boundary.

By Proposition 2.1 and the fact that D contains no charge, Theorem 2.1 is contradicted, and the proof is complete. □

2.1 Bounding the Number of Type I Equilibrium Points

We are now ready to put an upper bound on the number of type I equilibrium points of a configuration. This will be done through a series of smaller claims.

NOTATION - For a given configuration of charges and $c \in R$, let $P(c)$ denote the number of positive regions with respect to c in the potential field of the given configuration.

It is interesting to note that by Proposition 2.2, the $P(c)$ is always a finite positive integer, as long as the number of charges is finite. Since $c \in R$, it becomes easy to see P is a step function. We shall create a way to describe the "steps" of P.

Definition 2.2. *We say $P(c)$ **changes by** d **at** c if* $\lim_{\delta \to 0^+}[P(c+\delta) - P(c-\delta)] = d$.

Lemma 2.1. *Given any configuration of positive point charges in R^3. As c decreases, $P(c)$ cannot increase.*

Proof. Choose $c_1, c_2 \in R$ such that $c_1 > c_2$. Note that $\{p \in R^3 : V(p) > c_1\} \subset \{p \in R^3 : V(p) > c_2\}$. Therefore, any positive region of c_1 is a subset of a positive region of c_2.

Since $V(p) \to \infty$ at positive point charges, every point charge must be contained in some positive region of c_1. By Proposition 2.2, this would mean every positive region of c_2 contains at least one positive region of c_1.

By pigeon hole principle, since $P(c_1)$ pigeons completely fills $P(c_2)$ holes, we must have $P(c_2) \leq P(c_1)$. The proof is thus complete. □

Corollary 2.1. *Given positive point charges $\{p_1, ..., p_k\} \in R^3$ and $c \in R$, if P changes at c by d then the number of type I equilibrium points with potential c is no greater than d.*

Proof. Let $\{m_i\}$ be the set of equilibrium points with potential c.

Using algorithm presented with the special note in section 1.2, we should examine them as though they occured one after another. Without loss of generality, assume m_i occurs at potential c_i, where $c_1 > c_2 > c_3 > ...$ and each c_i is very close to c. Denote d_i as the amount P changes at c_i.

Since type I equilibrium points occur when multiple positive regions combine into one positive region, if m_i is a type I equilibrium point, P would decrease at c_i. In other words, if m_i is a type I equilibrium point, d_i is positive.

By Lemma 2.1, P cannot increase at any c_i. Thus, d and all d_i are non-negative integers.

Now notice that by definition, $d = \sum_i d_i$. Since $d, d_i \in N$, we conclude there can be at most d non-zero d_i. Therefore, there are at most d values of c_i that have type I equilibrium point occuring.

By our definition of c_i, only one equilibrium point occurs per c_i, and the corollary is proven. □

Theorem 2.3. *Given a configuration of k positive point charges, the number of type I equilibrium points cannot exceed $k-1$.*

Proof. By Proposition 2.2, $P(c)$ never exceeds k, even if c is very large.

Since the potential is continuous away from charges and the charges themselves have positive potential, we see there are no points in the potential field potential approaching $-\infty$. Thus, as $c \to -\infty$, all of R^3 becomes a single positive region (i.e. - $\lim_{c\to\infty} P(c) = 1$).

Since P is a step function, there exist countable number of c_i where P changes. If we let d_i denote the amount P changes at c_i, we have $k-1 \geq \sum_{i=1}^{\infty} d_i$.

By Corollary 2.1, we conclude that the number of type I equilibrium points does not exceed $k-1$. □

This is equal to the maximum number of type I equilibrium points proposed by Maxwell's Conjecture.

2.2 Further Study - Type II Equilibrium Points

While we have proven Maxwell's Conjecture on the number of type I equilibrium points, type II equilibrium points have much more complicated behavior. Unlike type I equilibrium points, their numbers have no obvious relation to the number of positive regions as c decreases.

We can still say many things about them, but to do so will require study of other concepts.

3 Location of Equilibrium Points

Before determining how many type II equilibrium points a configuration has, it is useful to try restricting the area in which we look for them. To do so, we will define a rather intuitive concept, put in more formal language.

Definition 3.1. *Let $\{p_1, ..., p_k\} \in R^3$ be a configuration of charges. The **convex hull** of*

$\{p_1, ..., p_k\}$ is the set of all $p \in R^3$ such that

$$p = \sum_{i=1}^{k} \alpha_i p_i$$

Where $\alpha_i \in R$ and $\alpha_i \geq 0$ $\forall i$ and $\sum_{i=1}^{k} \alpha_i = 1$.

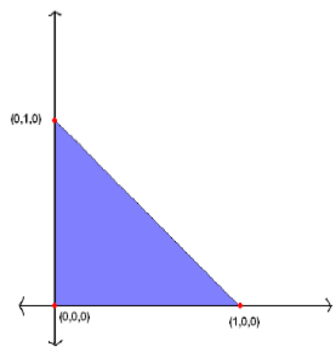

Figure 4
Convex Hull of (0,0,0), (1,0,0), and (0,1,0)
Cross section of at z=0

Graphically, the convex hull is all the points of the line, polygon, or polyhedron (depending on the configuration) formed by connecting the point charges. For example, the convex hull of given $\{p_1, p_2, p_3\} \in R^3$ would be the triangle with p_1, p_2, p_3 as its vertices. (See Figure 4)

We finish this section with a known property about convex hulls. For a proof of this theorem, see [2].

Theorem 3.1. *Given a configuration of positive point charges $\{p_1, ..., p_k\} \in R^3$, every equilibrium point lies in the convex hull of $\{p_1, ..., p_k\} \in R^3$.*

A more powerful version of this theorem will be presented in this paper. However, since the more powerful theorem will not be used to achieve any further result, we will present it at the end, in section 7.

4 Charges on a Plane

Definition 4.1. *We say that a configuration of charges* $\{p_1, ..., p_k\} \in R^3$ ***lies on a plane*** *if* $\exists$ *an isometry* $f : R^3 \to R^3$ *such that* $\forall p_i$*, the z-coordinate of* $f(p_i)$ *is 0.*

Our study of type II equilibrium points will be restricted to the case where all points in our configuration lie on a plane. Since we've used the concept of isometries in our definition, it is appropriate to note the relationship between isometries and the potential function.

Proposition 4.1. *Given a configuration of charges* $\{p_1, ..., p_k\} \in R^3$*, if* $f : R^3 \to R^3$ *is an isometry, then for any* $p \in R^3, V(p)$*, with respect to* $\{p_1, ..., p_k\}$*, is equal to* $V(f(p))$*, with respect to* $\{f(p_1), ..., f(p_k)\}$*.*

Proof. By Definition of Potential,

$$V(f(p)) = \sum_{i=1}^{k} \frac{q_i}{\mid f(p) - f(p_i) \mid}$$

By Definition of Isometry, $\mid f(p) - f(p_i) \mid = \mid p - p_i \mid$, and the proposition is proven. □

More specifically, this means that $p \in R^3$ is an equilibrium point of $\{p_1, ..., p_k\}$ if and only if for any isometry $f, f(p)$ is an equilibrium point of $\{f(p_1), ..., f(p_k)\}$.

Therefore, we can, without loss of generality, assume any configuration of charges that lies on a plane lies on the set $R^3_{z=0} = \{p \in R^3 :$ the z-coordinate of p is $0\}$, without worrying about a change in the number of equilibrium points.

We end this subsection with a direct corollary of theorem 3.1.

Corollary 4.1. *Given a configuration of positive point charges* $\{p_1, ..., p_k\} \in R^3_{z=0}$*, every equilibrium point of* $\{p_1, ..., p_k\}$ *also lies on* $R^3_{z=0}$*.*

4.1 Charges On an a Plane and R^2

As all charges and equilibrium points of a configuration lie in $R^3_{z=0}$, it becomes tempting to view them as elements of R^2. There are dangers to doing this, but we can go a certain distance with the idea.

Usually, the potential in R^2 is defined as $W(p) = \sum_{i=1}^{k} q_i \log \mid \frac{1}{p-p_i} \mid$, with the goal of making the potential harmonic. However, for our purposes, even though the charges and equilibrium points all lie in $R^3_{z=0}$, the potential field itself permeates throughout all of R^3, so we will use a definition closer to definition 1.1.

Definition 4.2. *Given a configuration of k point charges in 2-space, respectively with charges* $\{q_1, q_2, ..., q_k\}$ *and at locations* $\{p_1, p_2, ..., p_k\} \in R^2$ *, the* ***plane potential*** *of a given point* $p \in R^2$ *is defined as*

$$W(p) = \sum_{i=1}^{k} \frac{q_i}{\mid p - p_i \mid}$$

We say $p \in R^2$ *is an* ***equilibrium point*** *of* W *if* $\bigtriangledown W(p) = 0$.

Using the above definition, we see that many properties of V (3-dimensional potential) carry over to W (plane potential).

Proposition 4.2. *Given a configuration of point charges* $\{p_1, ..., p_k\} \in R^3_{z=0}$, $\frac{\partial}{\partial x} V(x, y, 0) = \frac{\partial}{\partial x} W(x, y)$ *and* $\frac{\partial}{\partial y} V(x, y, 0) = \frac{\partial}{\partial y} W(x, y)$ *for all* $x, y \in R$.

Proof. Letting each p_i be represented as $(x_i, y_i, 0)$, simple computation yields the following results:

$$\frac{\partial}{\partial x} V(x, y, 0) = \frac{\partial}{\partial x} W(x, y) = \sum_{i=1}^{k} \frac{-q_i(x - x_i)}{[(x - x_i)^2 + (y - y_i)^2]^{3/2}}$$

$$\frac{\partial}{\partial y} V(x, y, 0) = \frac{\partial}{\partial y} W(x, y) = \sum_{i=1}^{k} \frac{-q_i(y - y_i)}{[(x - x_i)^2 + (y - y_i)^2]^{3/2}}$$

□

Corollary 4.2. *Given a configuration of positive point charges in* $R^3_{z=0}$, *a point* $p = (x, y, z) \in R^3$ *is an equilibrium point of* V *if and only if* (x, y) *is an equilibrium point of* W.

Proof. From proposition 4.2, $\bigtriangledown V(x, y, 0) = 0 \iff \bigtriangledown W(x, y) = 0$, and the $\Leftarrow$ direction is immediately shown.

By corollary 4.1, $\bigtriangledown V(x, y, z) = 0 \Rightarrow z = 0$. Thus, $\bigtriangledown V(x, y, z) = 0 \Rightarrow \bigtriangledown W(x, y) = 0$. □

Proposition 4.3. *On any set in* R^2 *that has a non-empty interior and does not contain any point charges, the plane potential is subharmonic* ($\triangle W \geq 0$)*, continuous, and non-constant.*

Theorem 4.1 (Maximum Principle for Subharmonic Functions)**.** *Given any open set $D \subset R^3$, let u be a function that is subharmonic in D, continuous on $\bar{D}$, and non-constant. The maximum values of u on $\bar{D}$ are obtained on ∂D and nowhere else.*

Corollary 4.3. *The plane potential has no local maxima except at point charges.*

A proof of theorem 4.1 can be found in [6]. With these properties established, we can study configurations in R^2.

5 Type II Equilibrium Points on a Plane

Definition 5.1. *Given a region $P \subset R^2$, a point p is call an* ***internal point*** *of P if $p \notin P$ and there exists a closed path $l \subset P$ such that p is in the region bounded by l.*

A region S is called an ***internal bound*** *of P if S is connected, every point in S is an internal point of P, and S is not a subset of any region with the previous two properties.*

We call a region P ***internally bounded*** *if it has at least one internal bound.*

The notion of an internal bound of l in Def 5.1 is justified by the Jordan Curve Theorem. [1]

According to Maxwell, there are two sub-types of type II equilibrium points: those that produce internal bounds in positive regions and those that result from internal bounds of positive regions. In the context of the plane potential, this is simply the difference between a local minimum and a saddle point.

When making these claims, Maxwell generalized the concept of internal bound to R^3. This generalization will be addressed in section 5.1. For this section, however, we will only concern ourselves with positive regions in R^2.

Proposition 5.1. *Given a configuration of positive point charges in R^2, a point m is a local minimum of W if there exists $c \in R$ and a positive region P with respect to c such that m is an internal point of P. Furthermore, if a positive region P is internally bounded, it has at least one local minimum in each internal bound.*

Proof. Assume m is a local minimum. There thus exists $\epsilon' > 0$ such that for all $p \in B_{\epsilon'}(m)$, $W(p) \geq W(m)$. Consequently, any point $q \in B_{\epsilon'}(m)$, such that $W(q) = W(m)$, would also be a local minimum.

We stated in the first section of this paper that the number of equilibrium points is assumed to be finite. There thus exist a finite number of local minima in $B_{\epsilon'}(m)$. Let $\epsilon_1 = 1/2 \cdot \min\{\epsilon', q \in B_{\epsilon'}(m) : W(q) = q(m), q \neq m\}$.

Since positive point charges are clearly not local minima, and since R^2 is normal (theorem 2.2), there exists $\epsilon_2 > 0$ such that $B_{\epsilon_2}(m)$ contains no point charges.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and let $k = \inf\{W(p) :\mid p - m \mid = \epsilon\}$. By our choice of $\epsilon_1, k > W(m)$. Since W is continuous inside $B_\epsilon(m)$, then for any $c \in (W(m), k)$, the set $N = \{p : p \in B_\epsilon(m), W(p) < c\}$ contains points beside m.

Therefore, at potential $c \in (W(m), k)$, the set $B_\epsilon(m) - N$ would be part of a positive region, and would be internally bounded by negative region N, of which m is an element. ✓

The second part of the proposition is simpler. Assume positive region P (with respect to arbitrary c) is internally bounded by negative region N. By definition of internal bound, N is bounded. Since W itself is a bounded function, the absolute minimum of W in N is a local minimum in R^2. □

Notice that as c decreases, positive regions grow, and thus their internal bounds shrink. Since the potential is continuous, the internal bounds will eventually shrink to a single point, forming an equilibrium point. Equilibrium points that form in this manner are what we call equilibrium points that result from internal bounds.

Proposition 5.1 has shown us these equilibrium points are necessarily local minima.

Since by Corollary 4.3, W has no local maxima, every other type II equilibrium point is a saddle point.

Definition 5.2. *Given point charges $\{p_1, ..., p_k\} \in R^2$, a **type II saddle point** is an equilibrium point that is both a saddle point of W and a type II equilibrium point.*

Wheras type I equilibrium points are formed by interaction between multiple positive regions as c decreases, type II equilibrium points are formed entirely through a positive region's interaction with itself.

It then becomes appropriate to refer to positive regions P_1, P_2 respecting different potentials $c_1 < c_2$ as the same region as long as no type I equilibrium points occur between c_1 and c_2. One can imagine that as the potential decreases from c_2 to c_1, the region P_2 "morphs into" P_1.

Such a method of looking at positive regions will be important in denoting when internal bounds and type II equilibrium points occur.

Definition 5.3. *We say a positive region P* ***obtains object X*** *at potential c if P has object X at c but does not have object X as the potential approaches c from above.*

Theorem 5.1. *For any given $c \in R$, if a positive region in R^2 obtains a type II saddle point at c, it must also obtain an internal bound at c.*

Proof. Call the positive region P. Call the equilibrium point m.

Part 1

Our first goal is to prove P is internally bounded at c.

Since m is a saddle point, there must be a straight line segment l_1 going through m such that W obtains its minimum in l_1 at m. Since the potential at m is c, by definition of positive region, the end points of l_1 are in P.

Call these end points p_1, p_2. We claim there exists another path $l_2 \subset P$ connecting p_1, p_2 that doesn't touch m. If every path connecting p_1, p_2 must go through m, then the removal of m from P would cause P to be disconnected, and thus become two positive regions. This would make m a type I equilibrium point (see definition), which contradicts our hypothesis.

Consider L, the closed region bounded by paths l_1, l_2.

Since R^2 is normal, l_2 is closed and $m \notin l_2$, there must exist a ball of positive radius such that l_2 and $B(m)$ are disjoint. By our definition of l_1, we note that some subset of l_1 is a diameter of $B(m)$. Since the other path forming L, l_2, is entirely outside of $B(m)$, we see that $L \cap B(m)$ is a half-circle.

Since m is on the diameter of half-circle $L \cap B(m)$, we see that for any arbitrarily short line segment l_ϵ going through m, $l_\epsilon \cap (L \cap B(m))$ contains points besides m.

Assume $L \subset P$. This would mean that for any arbitrarily short l_ϵ going through m, l_ϵ will not achieve its maximum at m. Therefore, m is not a saddle point. This contradicts our hypothesis.

On the other hand, assume not all of L is contained in P. By their definitions, l_1 and l_2 are both in P, so any elements of L not contained in P must be in the interior of L. This makes P internally bounded. ✓

Part 2

Now we will prove this internal bound is obtained at c.

Since m is a saddle point, there must be at least one l_ϵ on which the potential achieves its maximum at m. (i.e. every point on l_ϵ except m is in a negative region with respect to c)

Since $l_1 \in P$, this is not the case on any subset of l_1. Thus, one of the endpoints of l_ϵ must be in the interior of L while the other one is in some other negative region N.

If we consider some $c^+ > c$ such that c^+ is very close to c, we see that every point on l_ϵ has potential less than c^+. This means the interior of L is connected to N, therefore no subset of L that internally bounds P for c^+.

We conclude the internal bound is obtained at c. ✓ □

Another way of wording the second part of theorem 5.1 is that type II saddle points “result in” internal bounds or “produce” internal bounds, in that whenever a type II saddle point is formed, an additional internal bound is formed with it.

Corollary 5.1. *Give a configuration of positive point charges in R^2, there cannot be more type II saddle points than local minima.*

Proof. By theorem 5.1, for every type II saddle point, there exists at least one internal bound forming at some c. By theorem 5.1, every internal bound contains at least one local minimum. Therefore, for every type II saddle point, there is at least one local minimum. □

5.1 Type II Equilibrium Points and R^3

While corollary 5.1 places some form of rule on the number of type II equilibrium points, in R^3, the potential function V is harmonic, so it is never appropriate to refer to its equilibrium points as local minima.

Fortunately, working off Maxwell's description of equilibrium points, the concept of internal bounds translates nicely from R^2 to R^3 for charges on a plane.

Definition 5.4. *Given a region $P \subset R^3$, a point p is call an **internal point** of P if $p \notin P$ and there exists a closed path $l \subset P$ with the following properties:*

i) p and all points on l lie on a plane by isometry f (See definition 4.1)

ii) In R^2, $I(f(p))$ is in the region bounded by $I(f(l))$, where I is the identity function from $R^3_{z=0}$ to R^2. (i.e. - $I(x, y, 0) = (x, y)$).

"Internal bounds" and "Internally bounded" are defined in R^3 in the same way they are defined in definition 5.4.

Proposition 5.2. *Given a positive configuration $\{p_1, ..., p_k\} \in R^3_{z=0}$, $p \in R^3_{z=0}$ is an internal point of positive region $P \subset R^3$ with respect to c if and only if $I(p) \in R^2$ is an internal point of $I(P \cap R^3_{z=0}) \subset R^2$.*

Proof. Assume p is an internal point of P but $I(p)$ is not an internal point of $I(P \cap R^3_{z=0})$.

Let $l \subset P$ be the path satisfying the conditions in definition 5.4.

Consider the path $l^* = \text{proj}_{R^3_{z=0}}(l) = \{(x, y, 0) \in R^3 : (x, y, z) \in l \text{ for some } z\}$. Since l is a closed path, l^* must also be a closed path. $I(l^*)$ also trivially contains $I(p)$ within its bounds.

Thus, since $I(p)$ is not supposed to be an internal point of $I(P \cap R^3_{z=0})$, $I(l^*)$ must not be a subset of $I(P \cap R^3_{z=0})$. That is to say, there exists some $(x, y, z) \in l$ such that $V(x, y, z) \geq c > W(\text{proj}_{R^3_{z=0}}(x, y, z)) = W(x, y)$. By their respective definitions, $W(x, y) = V(x, y, 0)$.

We thus arrive at $V(x, y, z) > V(x, y, 0)$ for some (x, y, z). Since each point charge has z-coordinate 0, if we denote each point charge as $p_i = (x_i, y_i, 0)$ this would mean:

$$\sum_{i=1}^{k} \frac{1}{\sqrt{(x-x_i)^2+(y-y_i)^2+z^2}} > \sum_{i=1}^{k} \frac{1}{\sqrt{(x-x_i)^2+(y-y_i)^2}}$$

for some non-zero z, which is impossible.

We conclude that if p is an internal point of P, $I(p)$ is an internal point of $I(P \cap R^3_{z=0})$. ✓

The other direction of this proof is trivial. ✓ □

Corollary 5.2. *Give a configuration of positive point charges $\{p_1, ..., p_k\} \in R^3$ lying on a plane, there cannot be more equilibrium points forming internal bounds than equilibrium points formed by internal bounds.*

Proof. Let f be the isometry taking $\{p_1, ..., p_k\}$ to $R^3_{z=0}$.

Let m be an equilibrium point formed by an internal bound. By propositions 5.1 and 5.2, this is true if and only if $I(f(m))$ is a local minimum of W with configuration $\{I(f(p_1)), ..., I(f(p_k))\}$.

By theorem 5.1, every other type II equilibrium point (i.e. - any such that $I(f(p))$ is a type II saddle point) forms an internal bound.

This corollary then follows from corollary 5.1. □

6 Proof of Maxwell's Conjecture for Specific Cases

With the studies we've done so far on equilibrium points, we can prove conjecture 1.1 for certain arrangements of positive point charges.

6.1 Point Charges on a Line

Definition 6.1. *Given a configuration of charges $\{p_1, ..., p_k\} \in R^3$, we say that this configuration **lies on a line** if $\exists$ an isometry $f : R^3 \to R^3$ such that $\forall p_i \in R^3$, the y-coordinate and z-coordinate of $f(p_i)$ is 0.*

Similarly to points on a plane, we will, without loss of generality, assume all our point charges lie on the x-axis. It is clear that all configurations that lie on a line also lie on a plane. Thus, any proofs from previous sections apply in this case as well.

Theorem 6.1. *Let $\{p_1, ..., p_k\} \in R^2$ be a configuration of positive point charges lying on a line. For any $c \in R$ and any positive region P with respect to c, P is not internally bounded.*

Proof. Without loss of generality, assume all point charges lie in the x-axis. Let N be an internal bound of P.

By proposition 5.1, N must contain an equilibrium point $m = (x_m, 0, 0)$ (specifically local minimum). By theorem 3.1, m must on the x-axis as well.

Since $m \in N$ is an internal point, there must exist a path $l \subset P$ such that $m \notin l$ but m is in the region bounded by l. Therefore, there exists $p = (x_p, y_p, 0) \in l$ such that $x_m = x_p$ and $y_p \neq 0$.

Since $p \in P$ and $m \notin P$, we must have $W(p) > W(m)$ by definition of positive region, giving us the following inequality:

$$\sum_{i=1}^{k} \frac{q_i}{\sqrt{(x_i - x_p)^2 + y_p^2}} > \sum_{i=1}^{k} \frac{q_i}{\mid x_i - x_p \mid}$$

Where $\{q_1, ..., q_k\}$ are the respective charges associated with $\{p_1, ..., p_k\}$. Since $y_p \neq 0$, the above inequality is impossible. ✓

With that contradiction, the proof follows. □

Theorem 6.2. *Let $\{p_1, ..., p_k\} \in R^3$ be a configuration of positive point charges lying on a line. This configuration has no more than $k - 1$ equilibrium points.*

Proof. By proposition 5.2 and theorem 6.1, $\{p_1, ..., p_k\} \in R^3$ has no internally bounded positive regions. By corollary 5.1, then, it has no type II equilibrium points. The proof follows from theorem 2.3, which puts the $k - 1$ bound on the number of type I equilibrium points. □

This proves Maxwell's Conjecture 1.1 for positive point charges lying on a line.

6.2 Case of Three Point Charges

It should not be difficult to see that any arrangement of three point charges lie on a plane. Thus, we can apply all our results from the previous sections.

Although this paper has no proof of conjecture 1.1 for three point charges, it has made progress barring a single condition.

Conjecture 6.1. *Given three positive point charges in R^2, W takes at most one local minimum.*

With the above conjecture, we can conclude using proposition 5.2 and theorem 6.1 that the configuration has at most two type II equilibrium points. Adding in the maximum of two type I equilibrium points from theorem 2.3, we arrive at a maximum of 4 equilibrium points, which is what was given by conjecture 1.1.

7 Location of Equilibrium Points, Part II

This section will contain a stronger version of theorem 4.1. While it was not used to prove further results in this paper, it may be useful for future studies of equilibrium points lying on a plane. To prove this theorem, we begin with a new concept.

Definition 7.1. *We call a function $f : R^3 \to R^3$, an* ***affine function*** *if for any set of $\{a_i\} \in R^3$ and $\{\alpha_i\} \in R$ such that $\sum_{i=1}^{k} \alpha_i = 1$, we have*

$$f(\sum_i \alpha_i a_i) = \sum_i \alpha_i f(a_i)$$

An affine function can graphically be thought of as any function that preserves straight lines and centers of mass. That is to say, if a set of points S form a straight line in the domain of affine function f with midpoint $s \in S$, then $f(S)$ is a straight line and $f(s)$ is the midpoint of $f(S)$.

Theorem 7.1. *Every isometry $f : R^n \to R^n$, with topologies defined by the Euclidean metric, is a composition of translations, rotations, and reflections.*

Theorem 7.2. *Every isometry $f : R^n \rightarrow R^n$, with topologies defined by the Euclidean metric, is affine.*

A proof of theorem 7.1 can be found in [5]. [7] shows that translations, rotations, and reflections are affine, and that compositions of affine functions are affine, leading us to conclude theorem 7.2 by theorem 7.1.

Theorem 7.2 will be important in proving properties of the next concept we'll define.

Definition 7.2. *Given $p \in R^3$, we define a* ***circle of radius*** ϵ ***centered at*** p*, denoted $O_\epsilon(p)$, as: $O_\epsilon(p) = \{q = (x_q, y_q, z_q) \in R^3 : z_q = 0, \mid p - q \mid < \epsilon\}$*

Definition 7.3. *Given a configuration of charges $\{p_1, ..., p_k\} \in R^3$ such that the charges lie on a plane, we call $p \in R^3$ a* ***face point*** *if $\exists$ isometry $f : R^3 \rightarrow R^3$ with the following properties:*

i) The z-coordinate of $f(p_i)$ is 0 $\forall i$.

ii) The z-coordinate of $f(p)$ is 0.

iii) $\exists\ \epsilon > 0$ such that $O_\epsilon(f(p)) \subset S$. Where S is the convex hull of $\{f(p_1), ..., f(p_k)\}$.

In many ways this idea is similar to the interior of a set in R^2. However, since our charges lie in 3-space, it was appropriate to define a new concept.

Proposition 7.1. *Given a configuration of charges $\{p_1, ..., p_k\}$, let $\{a_1, ..., a_n\}$ be elements of the convex hull of $\{p_1, ..., p_k\}$. The convex hull of $\{a_1, ..., a_n\}$ is a subset of the convex hull of $\{p_1, ..., p_k\}$.*

Proof. Let a be in the convex hull of $\{a_1, ..., a_n\}$.

$\Rightarrow$ By definition of convex hull, $a = \sum_{i=1}^{n} \alpha_i a_i$, where $\sum_{i=1}^{n} \alpha_i = 1$.

Since each a_i is in the convex hull of $\{p_1, ..., p_k\}$, we have $a = \sum_{i=1}^{n} \alpha_i (\sum_{j=1}^{k} \beta_{i,j} p_j) = \sum_{j=1}^{k} (\sum_{i=1}^{n} \alpha_i \beta_{i,j}) p_j$, where for every i, $\sum_{j=1}^{k} \beta_{i,j} = 1$.

Now, by properties of α and β, we note that

$$\sum_{j=1}^{k}(\sum_{i=1}^{n}\alpha_i\beta_{i,j}) = \sum_{i=1}^{n}\alpha_i(\sum_{j=1}^{k}\beta_{i,j}) = \sum_{i=1}^{n}\alpha_i \cdot 1 = 1.$$

Thus, we see that if we let $\gamma_j = \sum_{i=1}^{n}\alpha_i\beta_{i,j}$, then $a = \sum_{j=1}^{k}\gamma_j p_j$ and $\sum_{j=1}^{k}\gamma_j = 1$. We can then conclude any arbitrary a in the convex hull of $\{a_1, ..., a_n\}$ is also in the convex hull of $\{p_1, ..., p_k\}$. □

Proposition 7.2. *Given a configuration $\{p_1, ..., p_k\}$, let $\{a_1, ..., a_n\}$ be a set of face points of $\{p_1, ..., p_k\}$. All elements of the convex hull of $\{a_1, ..., a_n\}$ is a face point of $\{p_1, ..., p_k\}$.*

Proof. Since $\{a_1, ..., a_n\}$ are face points of $\{p_1, ..., p_k\}$, there must exist an isometry f and real numbers $\epsilon_1, ...\epsilon_n$ such that:

i) The z-coordinates of $\{f(a_1), ..., f(a_n)\}$ and $\{f(p_1), ..., f(p_k)\}$ are zero.

ii) $\forall\ i \in 1, ..., n, O_{\epsilon_i}(a_i) \in S$, where S is in the convex hull of $\{f(p_1), ..., f(p_k)\}$

Let ϵ =min$\{\epsilon_i\}$. Clearly, $\forall i, O_\epsilon(a_i) \in S$.

Let a be any element in the convex hull of $\{a_1, ..., a_n\}$. By definition of convex hull and since f is affine, we see that $f(a) = \sum_{i=1}^{n}\alpha_i f(a_i)$, where $\sum_{i=1}^{n}\alpha_i = 1$.

Since each $f(a_i)$ has z-coordinate of 0, $f(a)$ also has z-coordinate 0 by definition. We conclude that f satisfies the first and second conditions for a being a face point of $\{p_1, ..., p_k\}$. ✓

Let k be any element of R^3 such that $\mid k \mid < \epsilon$ and the z-coordinate of k is 0. Note that

$$f(a) + k = \sum_{i=1}^{n}\alpha_i f(a_i) + k \cdot \sum_{i=1}^{n}\alpha_i = \sum_{i=1}^{n}\alpha_i(f(a_i) + k)$$

Since each $O_\epsilon(a_i) \in S$, each $(f(a_i) + k) \in S$ as well. By Proposition 7.1, this means $f(a) + k \in S$, $\forall k$ as described. By definition of k, however, any element of $O_\epsilon(a)$ can be described as $f(a) + k$ for some k, so we may conclude $O_\epsilon(a) \in S$.

The last condition for a being a face point of $\{p_1, ..., p_k\}$ is thus satisfied by f. Since our a was an arbitrary element of the convex hull of $\{a_1, ..., a_n\}$, the lemma is proven. □

With these properties of face points established, we are ready to prove the main theorem of the section. This will be done through a series of lesser lemmas, with the following lemma being the most instrumental.

Lemma 7.1. *Given a configuration of charges* $\{p_1, ..., p_k\} \in {R^3}_{z=0}$. *If for any isometry* $f : {R^3}_{z=0} \to {R^3}_{z=0}$ *fixing the origin, the two sets* $R^3_{y\geq 0,z=0}$ *and* $R^3_{y<0,z=0}$ *both contain face points of* $\{f(p_1), ..., f(p_k)\}$, *then the origin is a face point of* $\{p_1, ..., p_k\}$.

Proof. **Part 1**

Before going into this proof, consider $R^3_{z=0}$ with one face point, $m_1 = (x_1, y_1, 0)$, in $R^3_{x>0,y>0,z=0}$ and and one face point, $m_2 = (x_2, 0, 0)$ on the negative x-axis.

Take a point $p = (x_p, y_p, 0)$ on the straight line segment connecting m_1 and m_2 and consider an infinitely long straight line $l_p \subset R^3_{z=0}$ containing both p and the origin. This line clearly contains all points $(x, y, 0) \in R^3_{z=0}$ where $\frac{y}{x} = \frac{y_p}{x_p}$.

By Lemma 3.11, if there exists a face point $r \in l_p$ such that the origin is between p and r on l_p, then the origin must also be a face point. Since $p \in R^3_{y>0,z=0}$ by choice of m_1 and m_2, this means that if any point in $\{(x, y, 0) \in R^3_{y<0,z=0} : \frac{y}{x} = \frac{y_p}{x_p}\}$ is a face point, then so is the origin.

By its definition, points on the straight line connecting m_1 and m_2 take the form $p = (x_p, \frac{y_1}{x_1 - x_2} \cdot (x_p - x_2), 0)$, where $x_p \in [x_1, x_2]$. By choice of m_1 and m_2, $\frac{y_p}{x_p}$ can take any value greater than $\frac{y_1}{x_1}$ (as x_p moves from x_1 to 0) as well as any non-positive value (as x_p moves from 0 to x_2). ✓

Part 2

This long but conceptually simple proof can now be given with a series of affine isometries. We will start by constructing the conditions to apply Part 1.

NOTATION - For the rest of the proof, let $F_k = f_k \circ ... \circ f_1$. Note that by Theorem 3.2, F_k is an isometry as long as each f_i is an isometry.

Assume the origin is not a face point. Take any face point of $\{p_1, ..., p_k\}$, call it a_1. Let f_1 be a rotation that takes a_1 to the positive y-axis. Note that all rotations are affine isometries by Theorem 3.1.

By hypothesis, there must exist face point a_0 such that $f_1(a_0) \in R^3_{y<0,z=0}$. Let $f_{2,\alpha}$ be a $\frac{\pi}{2}$ counterclockwise rotation around z-axis. This will put $(f_{2,\alpha} \circ f_1)(a_0)$ in either $R^3_{x>0,y>0,z=0}$ or $R^3_{x>0,y<0,z=0}$. If it is in $R^3_{x>0,y<0,z=0}$, let $f_{2,\beta}$ be a reflection on the y-axis. Otherwise, let $f_{2,\beta}$ be the identity. Let $f_2 = f_{2,\beta} \circ f_{2,\alpha}$; f_2 is isometry by Theorem 3.1 and 2.2.

Now note that $F_2(a_1)$ is on negative x-axis and $F_2(a_0)$ is in $R^3_{x>0,y>0,z=0}$. The conditions for Part 1 have thus been met. ✓

Part 3

NOTATION - For all functions F and points $p \in R^3$, let $F_x(p)$ denote the x-coordinate of $F(p)$, $F_y(p)$ the y-coordinate, and $F_z(p)$ the z-coordinate.

By hypothesis, there must exist face point a_2 such that $F_2(a_2) \in R^3_{y<0,z=0}$. By solution of Part 1, $0 < \frac{F_{2,y}(a_2)}{F_{2,x}(a_2)} < \frac{F_{2,y}(a_0)}{F_{2,x}(a_0)}$.

$\Rightarrow$The angle $F_2(a_2)$ forms with the negative x-axis is smaller than the angle $F_2(a_0)$ forms with the positive x-axis.

$\Rightarrow$If we let f_3 be a rotation putting $F_3(a_2)$ on negative x-axis, we'd have $0 < \frac{F_{3,y}(a_0)}{F_{3,x}(a_0)} < \frac{F_{2,y}(a_0)}{F_{2,x}(a_0)}$.

At this point, by the hypothesis, we must find an a_3 such that $F_3(a_2) \in R^3_{y<0,z=0}$. We then similarly construct f_4 that putting $F_4(a_3)$ on x-axis, and see that $0 < \frac{F_{4,y}(a_0)}{F_{4,x}(a_0)} < \frac{F_{3,y}(a_0)}{F_{3,x}(a_0)}$.

Thus, inductively, we see that as k increases, $\frac{F_{k,y}(a_0)}{F_{k,x}(a_0)}$ must strictly decrease and converge to 0. We thus conclude $\{F_k\}$ converges to a function $F_\infty = f_\infty \circ f_2 \circ f_1$, where f_∞ is a rotation putting $F(a_0)$ on positive x-axis.

Let g be a reflection on the y-axis and let $G = g \circ F_\infty$. Since $0 < \frac{F_{k,y}(a_k)}{F_{k,x}(a_k)} < \frac{F_{k,y}(a_0)}{F_{k,x}(a_0)}\ \forall k$, $\frac{F_{k,y}(a_0)}{F_{k,x}(a_0)} \to \frac{F_{\infty,y}(a_0)}{F_{\infty,x}(a_0)} = 0$ as k increases, and g is an isometry, we see that $\{\frac{G_y(a_k)}{G_x(a_k)}\} \to 0$ as k increases.

Lastly, noting $G(a_0)$ is on the negative x-axis and $G(a_k) \in R^3_{x>0,y>0,z=0}\ \forall k$, we apply Part 1 one last time to see there cannot exist any face points in $R^3_{y<0,z=0}$ unless the origin is a face point. This contradicts the hypothesis, and the theorem is proven. □

Lemma 7.2. *Let $\{p_1, ..., p_k\}$ be a configuration of charges. If $f(m)$ is a face point of $\{f(p_1), ..., f(p_k)\}$, then m is a face point of $\{p_1, ..., p_k\}$.*

Proof. If $f(m)$ is a face point of $\{f(p_1), ..., (p_k)\}$, then $\exists$ isometry g such that $g \circ f$ satisfies the conditions in Definition 3.14 when applied to $\{p_1, ..., p_k\}$. By Theorem 3.2, $g \circ f$ is an isometry, making m a face point. □

Lemma 7.3. *Given a configuration of charges $\{p_1, ..., p_k\} \in R^3$ such that the charges lie on a plane, each p_i is a limit point of the set of face points of $\{p_1, ..., p_k\}$.*

Proof. Let f be an isometry that brings $\{p_1, ..., p_k\}$ to $R^3_{z=0}$. Without loss of generality, we will show p_1 is a limit point of the face points of $\{p_1, ..., p_k\}$.

Since the given configuration does not lie on a line, there must exist p_i and p_j such that p_1, p_i, p_j do not lie on a line. We can thus conclude that the convex hull of $f(p_1), f(p_i), f(p_j)$ has a non-zero surface area.

Let $p_{i,n} = \frac{1}{n}p_1 + (1 - \frac{1}{n})p_i$ and $p_{j,n} = \frac{1}{n}p_1 + (1 - \frac{1}{n})p_j$ for $n \in N$. Since $p_{i,n}$ and $p_{j,n}$ lie on the straight lines connecting p_1, p_i and p_1, p_j respectively, we see that that convex hull of $f(p_1), f(p_{i,n}), f(p_{j,n})$ also has non-zero surface area $\forall n \in N$.

This means that each triplet $f(p_1), f(p_{i,n}), f(p_{j,n})$ can have a circle inscribed in its convex hull. By Definition 3.14, the preimage of the center of this circle, call it $p_{o,n}$, is a face point of $\{p_1, ..., p_k\}$.

As n increases, notice that $f(p_1), f(p_{i,n}), f(p_{j,n})$ get closer to each other, and thus the surface area of the convex hull monotonically decreases, converging to 0. Therefore, $d(f(p_{o,n}), f(p_1)) \rightarrow 0$ as n increases. Since f is isometric, we conclude that $d(p_{o,n}, p_1) \rightarrow 0$ as well.

We have thus shown that p_1 is a limit point of $\{p_{o,n}\}$, and the lemma is proven. □

We can now prove the main theorem.

Theorem 7.3. *Given a configuration of positive point charges $\{p_1, ..., p_k\} \in R^3$ such that the charges lie on a plane and not on a line, every equilibrium point on the potential field generated by $\{p_1, ..., p_k\}$ is a face point of $\{p_1, ..., p_k\}$.*

Proof. Assume $m \in R^3$ is an equilibrium point but not a face point.

NOTATION - For all functions F and points $p \in R^3$, let $F_x(p)$ denote the x-coordinate of $F(p)$, $F_y(p)$ the y-coordinate, and $F_z(p)$ the z-coordinate.

Since $\{p_1, ..., p_k\}$ lie on a plane, $\exists$ an isometry $g : R^3 \to R^3$ such that $\forall i \in 1, ..., k$, $g_z(p_i) = 0$.

Let $h : R^3 \to R^3$ be the translation defined by $h(a, b, c) = (a - g_x(m), b - g_y(m), c - g_z(m))$ $\forall (a, b, c) \in R^3$. By Theorems 2.1 and 2.2, the function $f = h \circ g$ is an isometry.

We shall now set up the conditions for Theorem 3.11. We first establish that $g_z(m)$ is 0.

Since m is an equilibrium point, $\bigtriangledown V(p) = 0$. Thus, by Proposition 3.1, $\bigtriangledown V(g(p)) = 0$, which in turn is true only if its z-coordinate is 0.

By Definiton of Potential, the z-coordinate of $\bigtriangledown V(g(m))$ is

$$\sum_{i=1}^{k} \frac{q_i(g_z(p_i) - g_z(m))}{[(g_x(p_i) - g_x(m))^2 + (g_y(p_i) - g_y(m))^2 + (g_z(p_i) - g_z(m))^2]^{(3/2)}}$$

Since each $g_z(p_i) = 0$ and each $q_i > 0$, we see that the above sum is equal to 0 iff $g_z(m) = 0$. ✓

Now that we've established $g_z(m) = 0$, we see, by the definitions of h and g, that $f_z(p_i) = g_z(p_i) = 0$ $\forall i \in 1, ..., k$. In other words, $\{f(p_1), ..., f(p_k)\} \in {R^3}_{z=0}$. ✓

Note that $f(m)$ is the origin. Since m is not a face point, by Proposition 3.11, $f(m)$ cannot be a face point. This gets us the conditions we need for Theorem 3.11. By Theorem 3.11, there exists an isometry r such that

$$U = \{r(p) \in {R^3}_{z=0} : \text{y-coordinate of } r(p) \geq 0\}$$

contains all the face points of $\{r(f(p_1)), ..., r(f(p_k))\}$.

Since U is closed, by Lemma 3.12, U must also contain $\{r(f(p_1)), ..., r(f(p_k))\}$.

By Definiton of Potential, the y-coordinate of $\bigtriangledown V((r \circ f)(m))$ is

$$\sum_{i=1}^{k} \frac{q_i((r \circ f)_y(p_i) - (r \circ f)_y(m))}{[\sum_{A=x,y,z} ((r \circ f)_A(p_i) - (r \circ f)_A(m))^2]^{(3/2)}}$$

By Proposition 3.1, since m is equilibrium point, the above sum must be 0. We already know $(r \circ f)_y(m) = 0$ since $f(m)$ is the the origin, $(r \circ f)_y(p_i) \geq 0$ $\forall i$ since they're in U, and

$q_i > 0$ is given. Thus, the above sum is 0 only if $(r \circ f)_y(p_i) = 0 \ \forall i$. This, however, would mean the p_i lie on a line, and our configuration was defined on a plane.

We thus arrive at a contradiction, and may conclude that m is an equilibrium point $\Rightarrow m$ is a face point. □

7.1 Consequences in R^2

Since theorem 7.3 regards points charges on a plane, it may potentially be more comfortable to work in R^2.

Corollary 7.1. *Given a configuration of positive point charges* $\{p_1, ...p_k\} \in R^2$ *that does not lie on a plane,* p *is an equilibrium point of* W *only if* p *is in the interior of the convex hull of* $\{p_1, ...p_k\}$*.*

Proof. For a convex hull S lying entirely on $R^3_{z=0}$,let S_0 be the projection of S onto R^2. The definition of a face point $(x, y, z) \in S$ corresponds exactly to the definition of an interior point $(x, y) \in S_0$. Proof then follows from theorem 7.3 and corollary 4.2 □

7.2 Charges not on a Plane

Cases where a configuration of charges do not lie on a plane have similar rules to Theorem 7.3. However, this paper did not intend to study point charges not lying on a plane. Thus, for the sake of brevity, rules pertaining to charges not lying on a plane will be given as conjectures.

Conjecture 7.1. *Given a configuration of charges* $\{p_1, ..., p_k\} \in R^3$ *such that the charges do not lie on a line or a plane, and every charge is positive, every equilibrium point on the potential field generated by* $\{p_1, ..., p_k\}$ *is an interior point of the convex hull of* $\{p_1, ..., p_k\}$*.*

The train of thoughts that would arrive at this conjecture is very similar to those which lead to theorem 7.3. A generalized version of lemma 7.1 would be especially instrumental.

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