Corona Number and Saturation in Hexagonal Mosaic Knots

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Abstract

Square mosaic knots have many applications in algebra, such as modeling quantum states. We continue the work of the previous REU cohort in extending mosaic knot theory to a theory of hexagonal mosaic knots, which are knots and links embedded in a plane tiling of regular hexagons. We define a new knot invariant, the corona number, by restricting the placement of tiles. We establish the corona number for knots of nine or fewer crossings, excluding 9_{16}. We also examine tile patches with a high number of link crossings, which we describe as saturated polygons. Considering patches of varying size and shape, we identify the number of components that are produced in these saturated polygons, with a particular focus on patches circumscribed by regular and irregular hexagons. Finally, we discuss open questions relating to the saturated polygons and bounds on the corona number.

1 Background

1.1 Knots and Links

Before we jump in to hexagonal mosaic knots, we need to know what a knot actually is. So how do we make a knot? Well, take a piece of string and tie it together in any fashion. Now glue the ends together and you have a knot! A formal definition is given below.

Definition 1. A *knot* is a closed curve in $\mathbb{R}^3$ that has no thickness and does not intersect itself.

Figures 1 and 2 show two examples of knots, the unknot (or the trivial knot) and a trefoil. A trefoil is the simplest nontrivial knot.

Figure 1: The unknot.

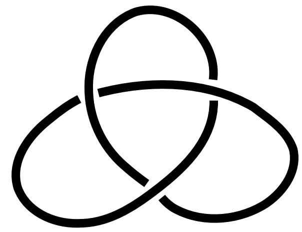


Figure 2: A Trefoil.

We can also use more than one piece of string, or have more than one curve.

Definition 2. A *link* is a collection of knots that are tangled together. Each "string" of a link L is called a *component* of L.

Knots can be seen as links with only one component. The Borromean rings, shown in the figure below, is a link with three components.

Figure 3: The Borromean rings

1.2 Reidemeister Moves

Note that there are infinitely many ways to draw a knot. We call drawings of knots and links, like the previous figures, knot *projections*. In a projection of a link, a *strand* is a portion of the link that goes from one undercrossing to another undercrossing, with only overcrossings in between. A strand of the knot on the right in the figure below can be seen in red. Note how our red strand begins and ends at an undercrossing, with two overcrossings in between.

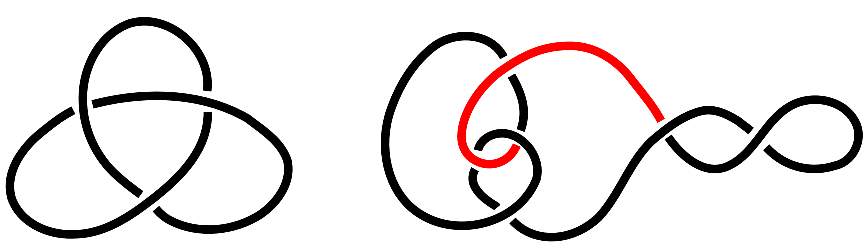


Figure 4: Two projections of a trefoil

But how do we know that the knot on the right in Figure **??** is in fact a trefoil? Well, in 1926 Kurt Reidemeister was able to prove the following theorem.

Theorem 1. Two projections are the same knot if and only if we can get from one projection to another through a series of Reidemeister Moves and planar isotopies.

There are three Reidemeister Moves, Type I, Type II, and Type III.

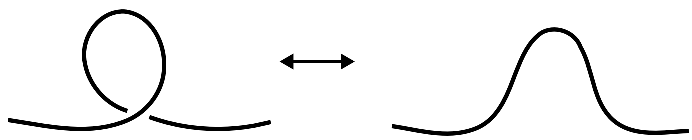


Figure 5: Type I Reidemeister Move.

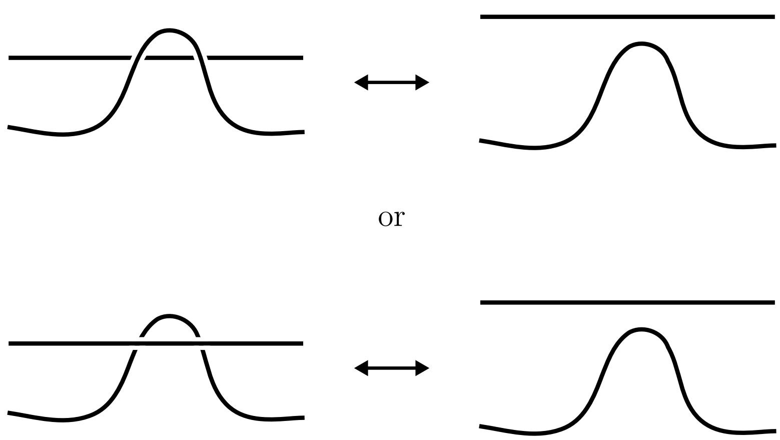


Figure 6: Type II Reidemeister Move.

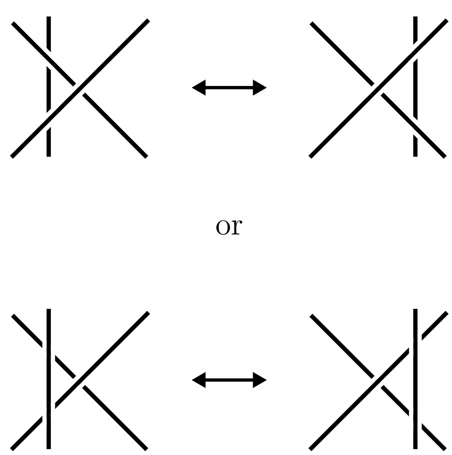


Figure 7: Type III Reidemeister Move.

Since knot theory is a subset of topology, then performing planar isotopies, or continuous deformations in the projection plane, on a knot does not affect the knot.

Figure 8: Planar Isotopy.

It is left as an exercise to the reader to use Reidemeister moves and planar isotopies to show that the knot on the right in Figure **??** is in fact a trefoil.

1.3 Knot Invariants

While Reidemeister moves can tell us if two projections are the same knot, it is difficult to use Reidemeister moves to show that two projections are distinct knots. Even if we are unable to get from knot K to knot L using thousands of different combinations of Reidemeister moves, that does not mean that K and L are distinct. Maybe we simply have not found the correct sequence of Reidemeister moves. So, we need a stronger tool to let us know when two knots or links are distinct.

Definition 3. A *knot invariant* is a characteristic of a knot (or link) K that does not depend on the projection of K. I.e., performing Reidemeister moves does not change the value of the knot invariant.

For example, tricolorability is a knot invariant. We say that a knot (or link) K is *tricolorable* if the following conditions hold:

(1) Each strand of K must be one of three colors

(2) At each crossing c, the three strands that meet at c must be either all the same color, or all distinct colors

(3) At least two colors must be used

Below we can see that the trefoil is tricolorable. Since tricolorability is a knot invariant, both projections of the trefoil seen in Figure **??** are tricolorable.

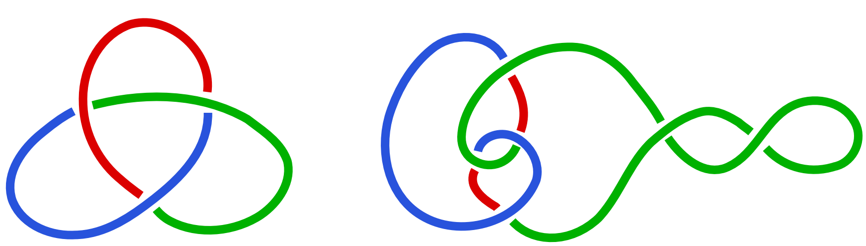


Figure 9: Two tricolored projections of the trefoil

Recall that for a knot K to be tricolorable, we must use at least two colors to color all the strands of K. Since the unknot only has one strand, we can use at most one color. Therefore the unknot is not tricolorable. Since the trefoil is tricolorable, and the unknot is not, we know that the trefoil is distinct from the unknot.

Tricolorability is only one example of several existing knot invariants, but researchers have yet to find THE knot invariant, i.e., a knot invariant ϕ such that two knots K, L are the same if and only if $\phi(K) = \phi(L)$. For each knot invariant that is currently known, we can find two distinct knots with the same knot invariant value.

1.4 Hexagonal Mosaic Knots

All of the definitions and concepts that we have covered so far apply to knot theory in general, and therefore they apply to the theory of hexagonal mosaic knots as well.

Definition 4. A *hexagonal mosaic knot* is a knot or a link embedded in a plane tiling of regular hexagons.

In Figure **??**, we have the 24 distinct tiles we will be using to construct hexagonal mosaic knots and links. These tiles are distinct up to symmetries of a regular hexagon, according to the three properties given below. We will refer to these hexagonal mosaic tiles as hextiles.

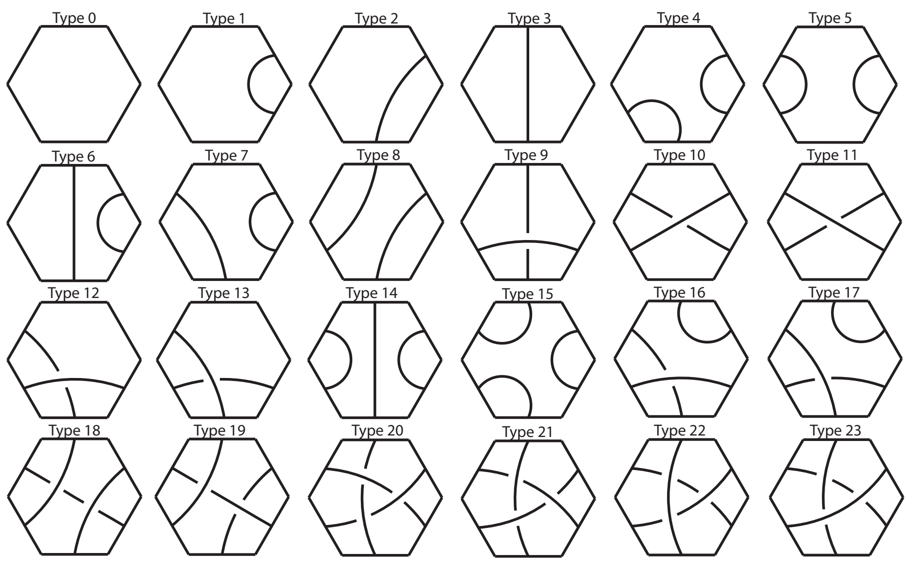


Figure 10: The set of 24 hextiles distinct up to rotation and reflection

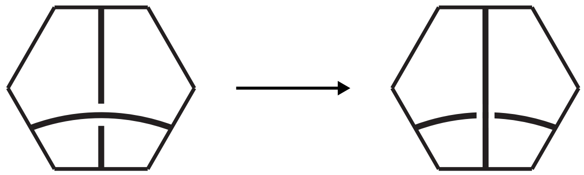
Each hextile obeys the following three properties:

(1) No curve intersects itself

(2) No curve intersects another curve more than once

(3) There is at most one curve per each edge of the hextile

Remark. To flip a tile over a line of symmetry, let overcrossings become undercrossings and vice versa, as if the strands on the tile were made of string and had depth to them. For example, flipping the title T_9 over the vertical line of symmetry gives



We will refer to a projection of a knot or link K using these hextiles as a *diagram* of K.

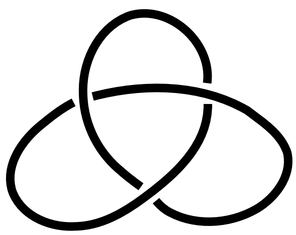


Figure 11: A normal projection of the trefoil.

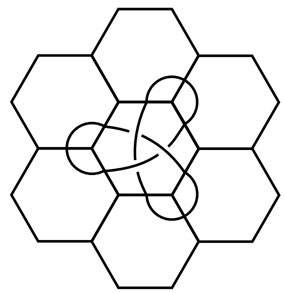


Figure 12: A diagram of the trefoil with 7 hextiles.

Definition 5. The point of intersection between a strand on a hextile and the edge of a hextile is called a *connection point*. A diagram is said to be *suitably connected* if the connection point of each hextile touches a connection point of its adjacent tiles.

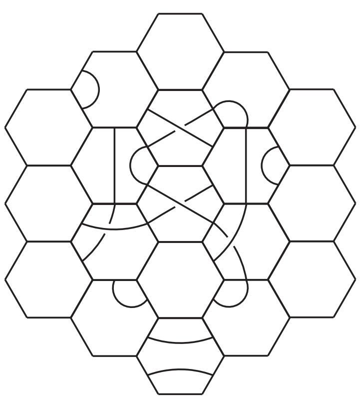


Figure 13: Not a suitably connected diagram.

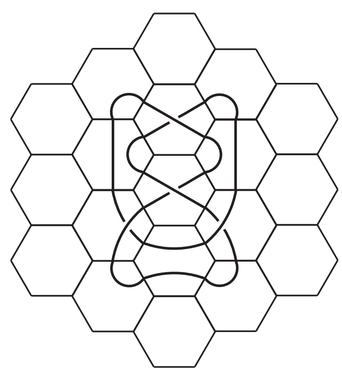


Figure 14: A suitably connected diagram of the figure eight knot.

The 2016 REU cohort at UW Washington proposed a new knot invariant, the *hextile number* of a hexagonal mosaic knot, or the minimal number of tiles necessary to create a knot diagram on hexagonal tiling [2]. The hextile number is a knot invariant that is analogous to the *tile number* in square mosaic knot theory [4]. Both the hextile number and the tile number of a knot measure the complexity of the knot; they measure how efficiently a knot takes up space. We define a new knot invariant, the corona number, that also measures the space-efficiency of a knot or link. But first, we need to understand the notion of a corona, which we define recursively.

Definition 6. Let the *zero-th corona*, or C_0, be a single hexagonal tile. For $n \geq 1$, define the *nth corona*, or C_n, to be the set of all hexagonal tiles T that are adjacent to C_{n-1} such that $T \notin C_{n-2}$.

In Figure **??**, C_0, C_1, C_2, C_3, C_4 are shaded blue, red, orange, green, and purple, respectively. We can also look at the union of distinct coronae. We say a *board of size n* is the set of tiles

$$S = \bigcup_{i=0}^{n} C_i.$$

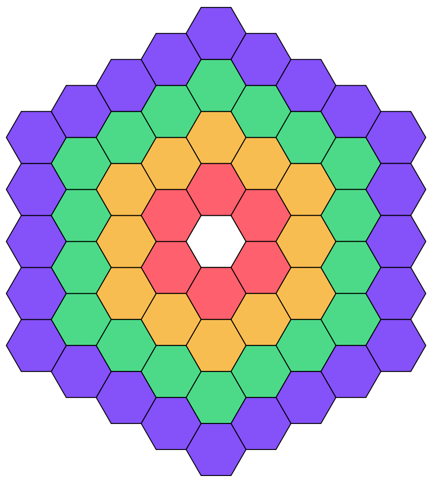


Figure 15: A board of size 4.

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Now that we understand a board of size n, we can move on to corona number.

Definition 7. The *corona number* of a knot (or link) K, denoted $\varkappa(K)$, is the least board size that permits a diagram of K. I.e., $\varkappa(K) = n$ if a board of size n permits a diagram of K, and no diagram of K fits on a board of size m for $0 \leq m < n$.

Just as the hextile number is analogous to the tile number, the corona number is analogous to the *mosaic number*, originally developed by Kauffman and Lomanaco [6]. Later on, we will show that the trefoil in Figure **??** has corona number 1. How many other knots will have corona number 1? What is the maximum number of crossings we can fit on a board of size n? We will explore these questions throughout our paper.

2 Corona Number

Throughout this section, we will establish the corona number for several lower-crossing knots. There are two parts to establishing the corona number of a knot K. To prove that $\varkappa(K) = n$, we need to not only show that a diagram of K fits on a board of size n, but we also need to show that a smaller sized board does not permit a diagram of K. The second part of establishing the corona number can be tricky. We can do this by taking advantage of other knot invariants that exist in classical knot theory, like the crossing number.

Definition 8. The *crossing number* of a knot (or link) K, denoted $c(K)$, is the least number of crossings in any projection of a knot. I.e. $c(K) = n$ if n is the smallest nonnegative integer such that any projection of K has at least n crossings.

Before we utilize the crossing number, note that the only tile permitted on a suitably connected board of size zero is T_0, or the empty tile, so we know that $\varkappa(K) \geq 1$ for any knot or link K.

Theorem 2. Let K be either the unknot or a trefoil knot. Then $\varkappa(K) = 1$.

Proof. Since $\varkappa(K) \geq 1$, it is sufficient to show that the unknot and the trefoil can be drawn on a board of size 1, as shown in the figures below.

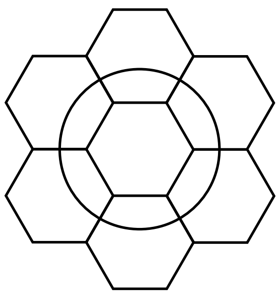


Figure 16: A diagram of the unknot on a board of size 1.

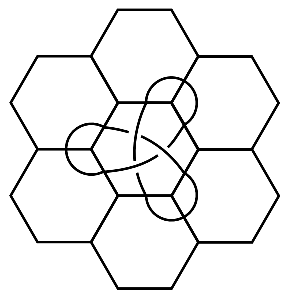


Figure 17: A diagram of the trefoil on a board of size 1.

□

Lemma 1. If K is a knot or link such that $c(K) \geq 4$, then $\varkappa(K) \geq 2$.

Proof. Let K be such a knot. It suffices to show that a board of size 1 does not permit a diagram of K. Note that any crossing tile has at least four distinct connection points. Moreover, on a board of size 1, C_1 is the outermost corona, so any tile T in C_1 has three edges that border the infinite outermost region and are therefore not contiguous with any other tile. In Figure **??**, we can see these edges highlighted in blue for each tile in C_1.

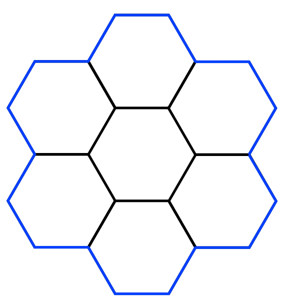


Figure 18: A board of size 1.

Since a diagram of K must be suitably connected, we cannot have any connection points on the outer blue edges. Hence, if T is on C_1, then T has at most three connection points, and is therefore not a crossing tile. This implies that the center tile can be the only crossing tile on a board of size 1. Since each hextile has at most three crossings, a suitably connected board of size 1 has at most three crossings. Since $c(K) \geq 4 > 3$, then a board of size 1 does not permit a diagram of K. $\square$

Theorem 3. Let K be a knot such that $4 \leq c(K) \leq 9$ and $K \neq 9_{16}$. Then $\varkappa(K) = 2$.

Proof. By Lemma **??**, we know that $\varkappa(K) \geq 2$. The appendix shows that a board of size 2 permits a diagram of K, therefore $\varkappa(K) = 2$. $\square$

3 Saturation

We will expand on the work of Howards and Kobin and their notion of "saturation" [5]. The main goal of saturation is to use as many crossings as possible in a certain patch of tiles. In the square mosaic case, that meant using tiles with one crossing. In our case, we will use hextiles with three crossings. We want to know what happens when we saturate these tile patches. Do we get a knot or a link? If we get a link, how many components will it have?

Definition 9. Let P be a polygon that is constructed from hextiles. The *perimeter* of P is the set of exterior tiles of P that border the infinite outermost region. The *inner board* of P is the set of remaining interior tiles.

In the figure below, we have a parallelogram P constructed from hextiles. The perimeter of P is shaded in yellow and the inner board of P is shaded in purple.

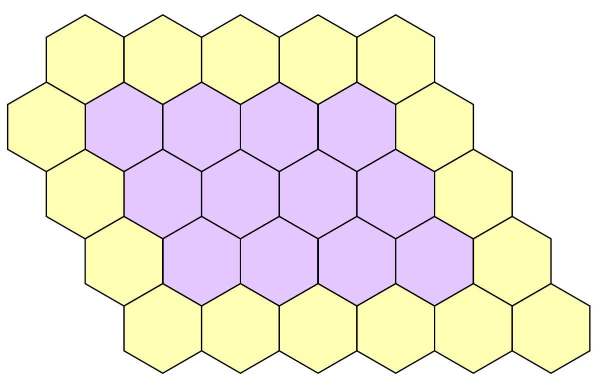
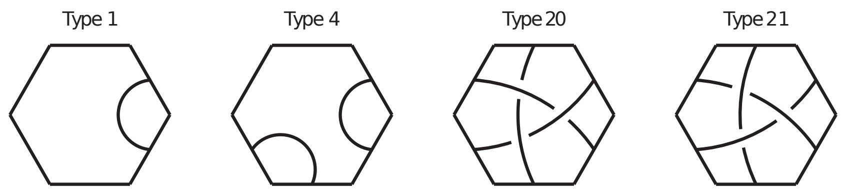


Figure 19: The perimeter and inner board of a parallelogram P.

Definition 10. Given a polygon P constructed from hextiles, we say P is *saturated* if the following conditions are satisfied:

1) Every hextile on the inner board of P is either T_{20} or T_{21}

2) The hextiles on the perimeter of P suitably connect the board using only T_1 and T_4.



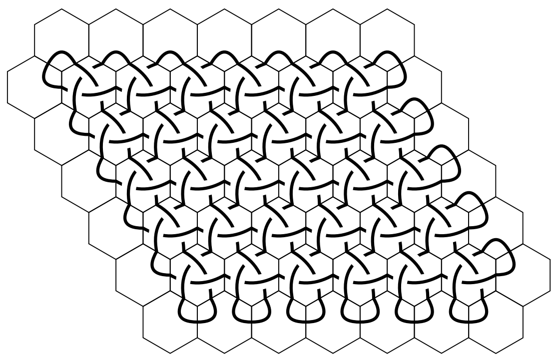


Figure 20: A saturated parallelogram.

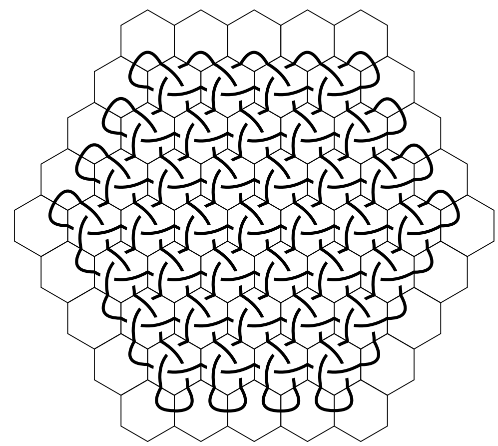


Figure 21: A saturated board of size 4.

Although T_{22} and T_{23} are also hextiles with three crossings, note that the crossings are not alternating; both tiles have a strand that has two consecutive over crossings, and another strand that has two consecutive under crossings. We want the crossings to be alternating because that minimizes the potential for a Reidemeister move to undo a crossing. We also want to keep this notion of saturation analogous to saturation in the square mosaic case, which does not permit crossings on the perimeter of an $n \times n$ board [5]. Therefore we define a saturated polygon to not allow crossing tiles on the perimeter and non-perimeter tiles to contain three crossings. It should be noted that this definition does not construct links with a maximum number of crossings possible since one can create suitably connected diagrams on polygons with crossings on perimeter tiles. With our definition, once the inner board (non-perimeter tiles) is placed, we can suitably connect the board in two ways.

A given edge on the perimeter of a saturated polygon P will consist either entirely of T_1 hextiles, or entirely of T_4 hextiles, as seen in the figure below.

Figure 22

As we traverse the perimeter of a saturated polygon, we will pass from edge to edge of the polygon. We call a hextile on two edges of the polygon a *corner tile*. Since we focus on hexagons in this paper, we show that if an edge e contains only T_4 tiles or only T_1 tiles (See

Cases A and B in Figure ??), then adjacent edges will contain T_1 tiles or T_4 tiles respectively. Since T_i is on the perimeter of P, then T_i is either T_1 or T_4. Since P is a saturated polygon, the edges highlighted in blue will have a connection point, so T_i has at least two connection points in both cases. Moreover, the edges highlighted in green border the infinite outermost region and therefore do not have any connection points. Hence T_i has at most four connection points in both cases.

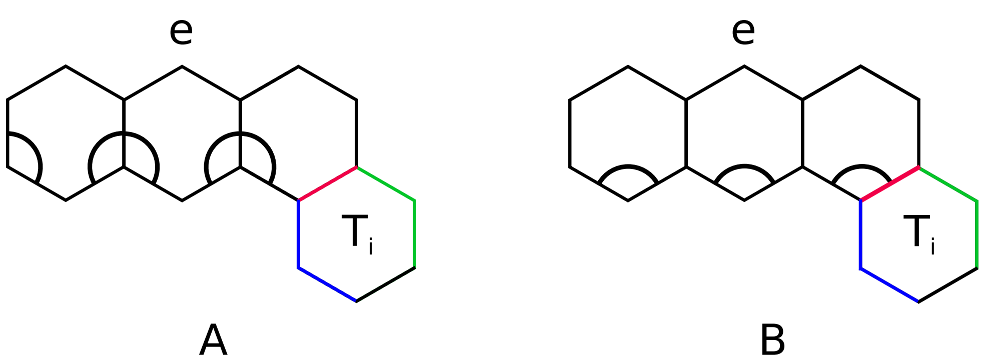


Figure 23

Note that for Case A, there is no connection point on the edge highlighted in red, so T_i has at most three connection points. However, there must be an even amount of connection points, so T_i has exactly two connection points, which implies that $T_i = T_1$. For Case B, we do have a connection point on the edge highlighted in red, so T_i has at least three connection points. Again, there must be an even amount of connection points, so T_i has exactly four connection points, which implies that $T_i = T_4$. Hence Case A and Case B will look like the Figure ??.

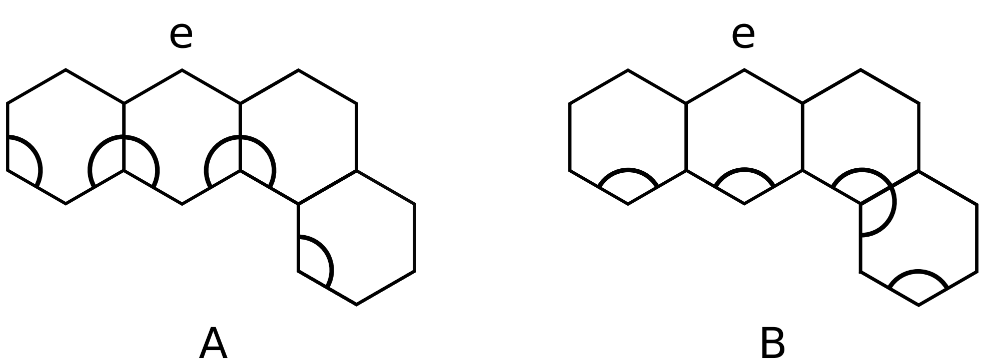


Figure 24

Thus hexagonal edges will alternate between T_1 and T_4 tiles as you traverse the perimeter of the hexagon.

While the 2016 REU cohort focused on saturated parallelograms [2], we will focus on saturated hexagons, both regular and irregular.

Definition 11. For $m, n \in \mathbb{Z}^+$, we say that H is an $m \times n$ *saturated stretched hexagon* if H is a saturated hexagon such that

1) Four sides of H have length m.

2) Two sides of H have length n.

3) The sides of length n are opposite each other.

4) H is oriented so that the sides of length n are horizontal, i.e. the top and bottom edges of H.

5) The top perimeter edge of H consists of T_1 hextiles and the bottom perimeter edge of H consists of T_4 hextiles.

Our notion of side length is illustrated in the figure below.

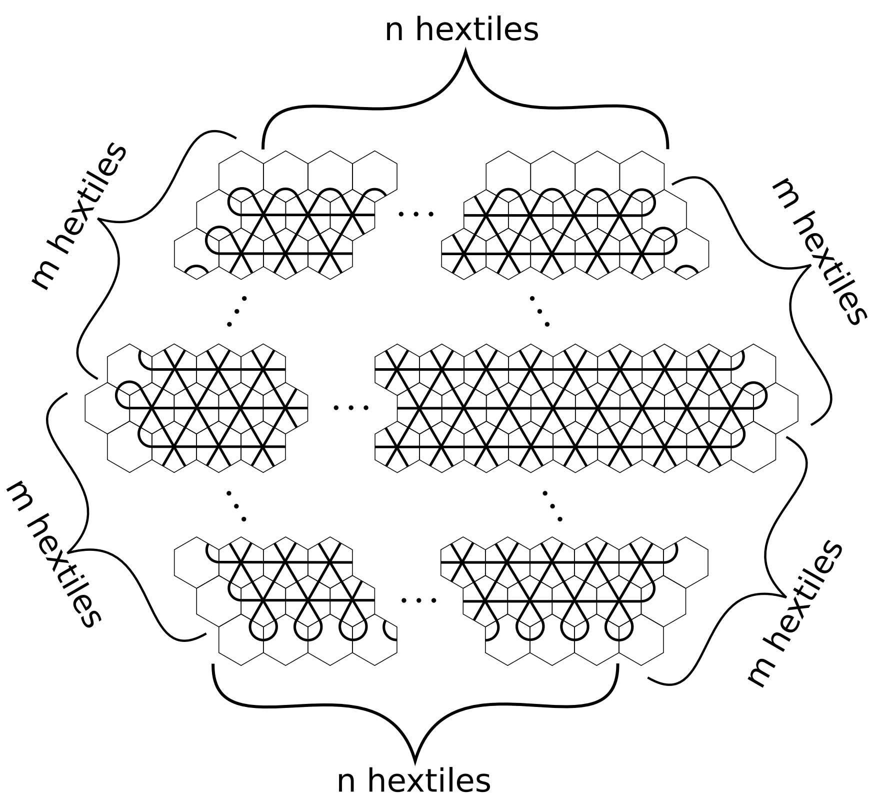


Figure 25: An $m \times n$ saturated stretched hexagon.

The crossings in Figure **??** are drawn ambiguously to show that we can use any rotation of T_{20} or T_{21} to saturate H. Note that if we want to saturate H so that the top perimeter tiles consist of T_4 hextiles and the bottom perimeter tiles consist of T_1 hextiles, we can simply rotate H by 180°. Therefore no generality is lost when H is oriented so that the top perimeter tiles consist of T_1 hextiles. Figure **??** shows a 3×6 saturated stretched hexagon that is saturated using only one orientation of T_{20}, while the 6×3 saturated stretched hexagon in Figure **??** contains different rotations of both T_{20} and T_{21}.

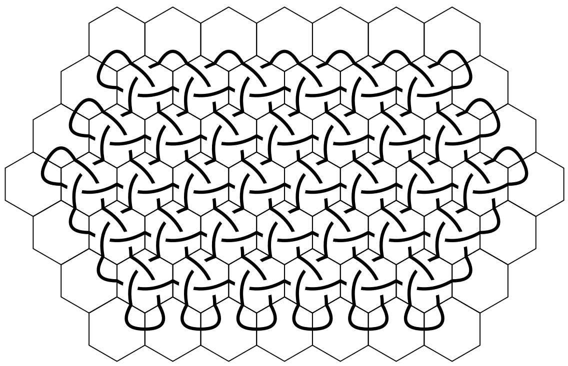


Figure 26: A 3×6 saturated stretched hexagon.

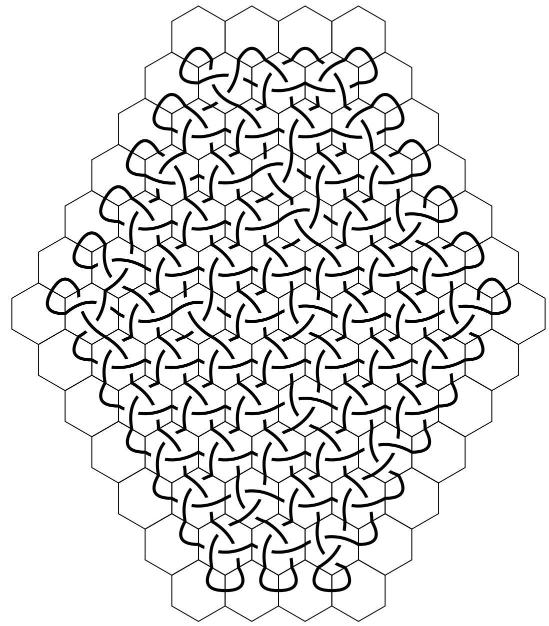


Figure 27: A 6×3 saturated stretched hexagon

When looking at a saturated polygon, we are interested in counting the number of components of the link that results from the saturated diagram. If one places a finger on an arbitrary point of the knot (or link) in the diagram, it is possible to follow the component as it "bounces" around the polygon (like a billiard ball on a billiard table) and goes back to the starting point. We will utilize a hexagonal coordinate system to keep track of the "bouncing" of each individual component.

Our hexagonal coordinate system will have two axes that intersect at a 60° angle, as seen in Figure **??**, and each hextile will represent exactly one point, with integer coordinates, in our hexagonal coordinate system.

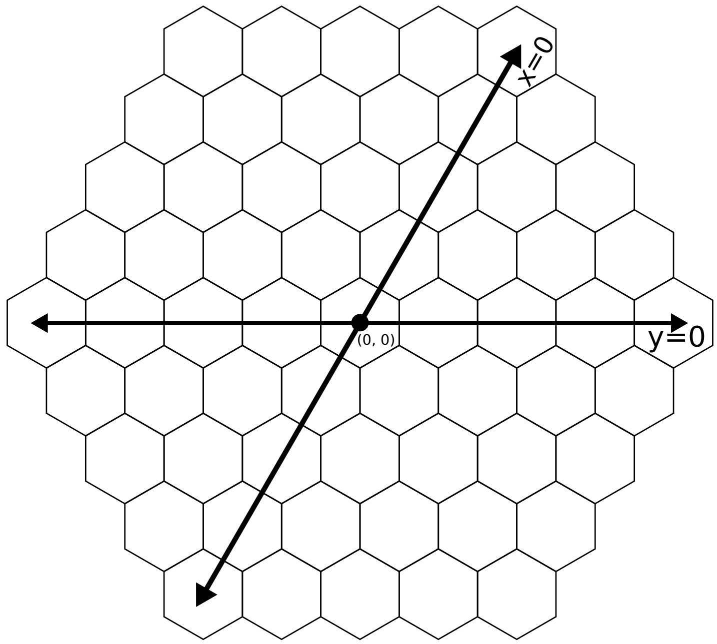


Figure 28: Hexagonal Coordinate System

Superimpose the hexagonal coordinate system on a graph of $\mathbb{R}^2$ with Cartesian coordinates, lining up the origins and the x-axes for both coordinate systems. Notice that the y-axis for the hexagonal coordinate system corresponds to the line $y = \frac{\sqrt{3}}{2}x$ in $\mathbb{R}^2$. Therefore, in $\mathbb{R}^2$, two unit vectors on the hexagonal coordinate system axes are $(1, 0)$ and $(1/2, \sqrt{3}/2)$. Let the point $k(0, 1) + l(1/2, \sqrt{3}/2) \in \mathbb{R}^2$, $k, l \in \mathbb{Z}$ correspond to the point (k, l) in the hexagonal coordinate system. Formally, we can convert hexagonal coordinates to Cartesian coordinates using the function ϕ, defined below.

$$\begin{aligned} \phi : \mathbb{Z} \times \mathbb{Z} &\to \mathbb{R}^2 \\ (k, l) &\mapsto k(0, 1) + l(1/2, \sqrt{3}/2), \end{aligned}$$

As seen in Figure **??**, points are defined at the centers of hextiles.

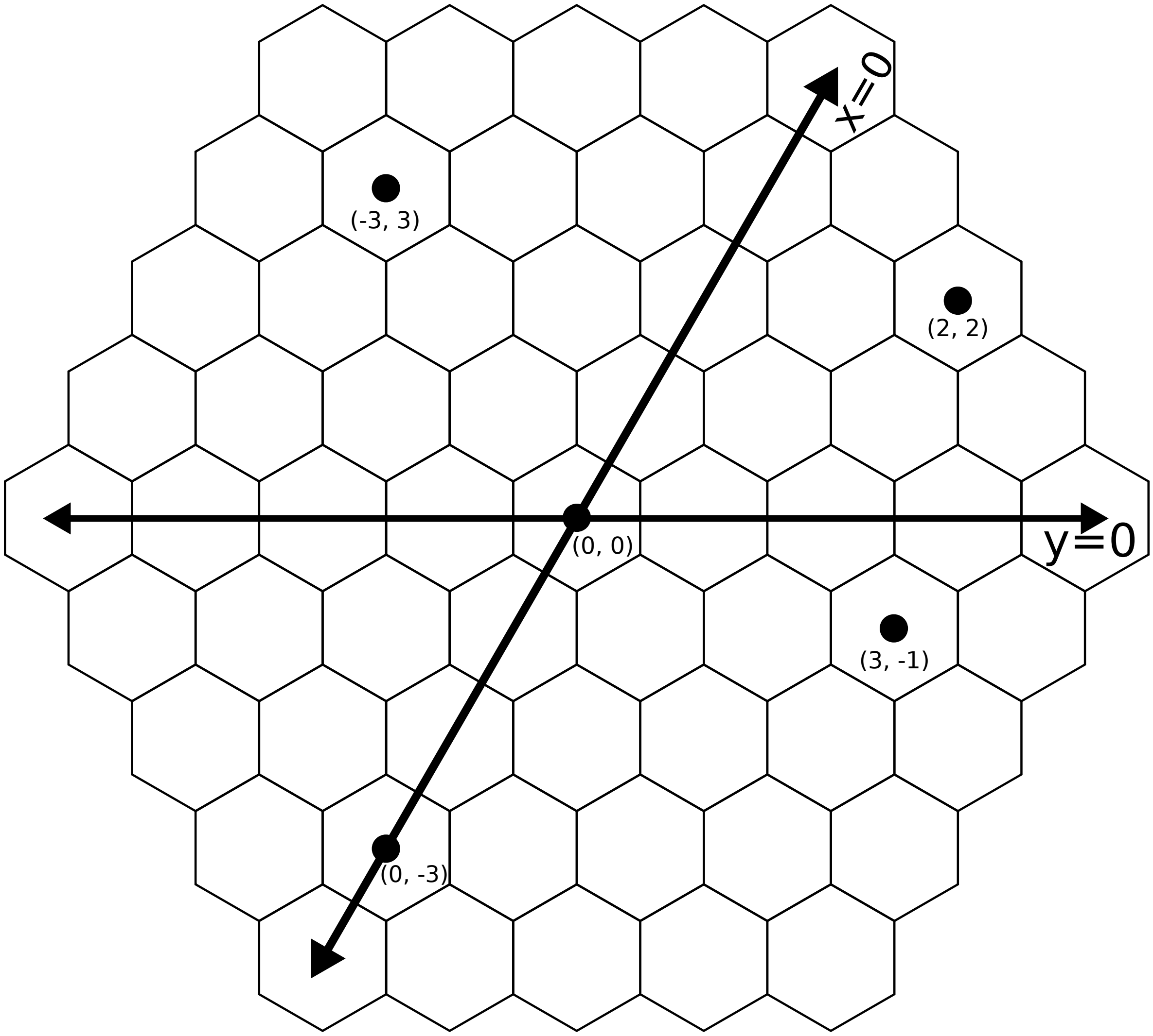


Figure 29: Hexagonal Coordinate System.

Lines of the form $x = k, y = j$, and $x + y = i$ will be pivotal moving forward. The origin and the point $(-3, 3)$ are both on the line $x + y = 0$.

This hexagonal coordinate system will help keep track of individual components as they "bounce" around the perimeter of an $m \times n$ saturated hexagon H. What exactly do we mean by "bounce"? Recall that the perimeter edges of H will either consist entirely of T_1 hextiles or entirely of T_4 hextiles. First consider an edge that consists of T_1 tiles. The component highlighted in red in Figure **??** travels along the line $x = j$, "bounces" off the line $y = k$ at the point (j, k), and continues to travel along the line $x + y = j + k$.

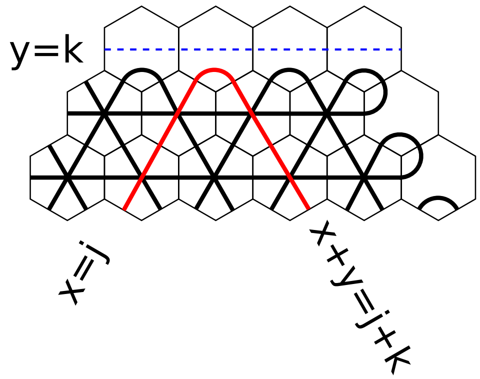


Figure 30

Now consider an edge that consists entirely of T_4 hextiles. Let C be the component highlighted in red in Figure ??. Just as before, C travels along the line $x = j$ and bounces off the line $y = k$ at the point (j, k). However, the Reidemeister 1 move that occurs among the T_4 hextiles causes C to behave differently from before. In this case, C travels along the line $y = k$ and bounces off the line $x = j + 1$ at the point $(j + 1, k)$, and then continues to travel along the line $x + y = j + k + l$.

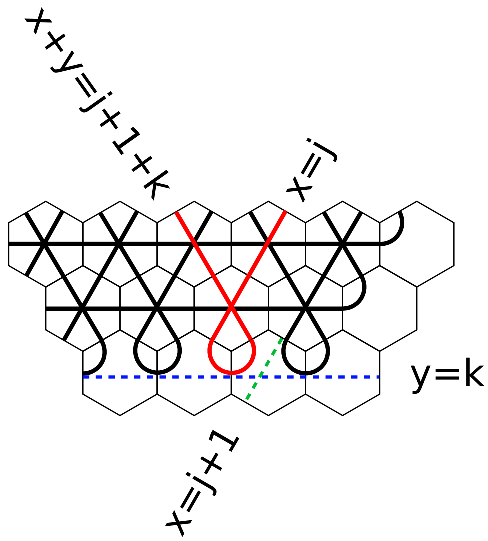


Figure 31

Components bounce off the lateral edges of H in a similar fashion as the two cases given above.

Theorem 4. Let H be an $m \times n$ saturated stretched hexagon. Then the link produced by H has exactly m components.

Proof. Begin with an $m \times n$ saturated stretched hexagon H, as in Figure ??. Place the origin at the left central corner so that the corners of H are $(0, 0), (0, m), (n, m), (n + m, 0), (n + m, -m)$, and $(m, -m)$ as they are read clockwise. We will describe the components of the link created by H; arguments will depend on the parity of n as well as the relationship between n and m.

1. Suppose n is even.

   Let T_j be the component of link containing segments of the lines $x = j$ or $x+y = m+j$. Notice that for $j = -m+1, -m+2, \ldots, 0$, T_j contains segments of $x + y = m + j$ (as shown in blue in Figure ??), but T_j does not contain segments of $x = j$, since $x = j$ is not a strand contained in H. For $j = 1, 2, \ldots, n-1$, T_j contains segments of both $x = j$ and $x+y = m+j$, as shown in red in Figure ??. Finally, for $j = n, n+1, \ldots, n+m-1$, T_j contains segments of $x = j$ (as shown in green in Figure ??), but T_j does not contain segments of $x + y = j + m$ since $x + y = j + m$ is not a strand contained in H.

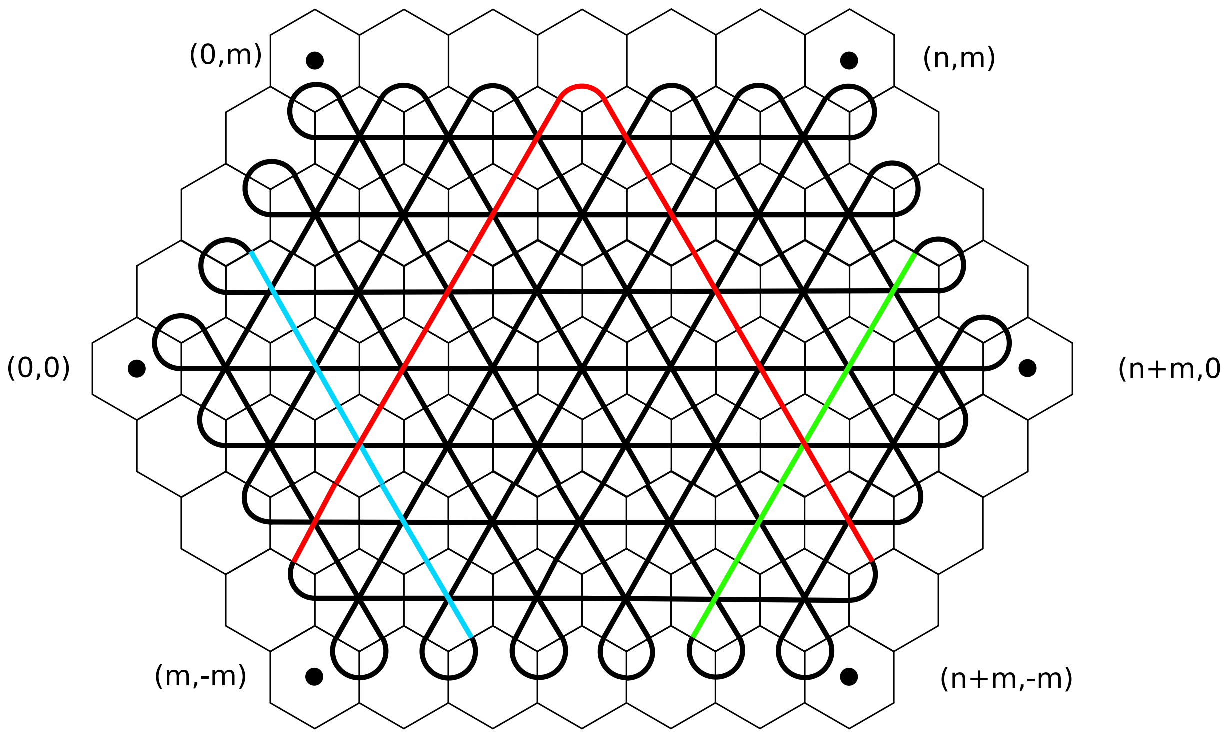


Figure 32: A 4×6 saturated stretched hexagon H.

   Note that $T_j = T_k$ if $j \mod (2m-1) = k \mod (2m-1)$. Let $\overline{j}$ denote $j \mod (2m-1)$. Then $T_j = T_k$ if $\overline{j} = \overline{k}$. Indeed, starting at the point (j, m) if we trace T_j to the left using the first table below or to the right using the second table below, we see that $T_j = T_{j-(2m-1)} = T_{j+(2m-1)}$. Repeated use of these tables yields the desired result.

Step	Follow along line	Until intersects line	Intersection point
L1	$x = j$	$y = -m$	$(j, -m)$
L2	$y = -m$	$x = j + 1$	$(j + 1, -m)$
L3	$x + y = j - m + 1$	$y = m$	$(j - (2m - 1), m)$

Step	Follow along line	Until intersects line	Intersection point
R1	$x + y = j + m$	$y = -m$	$(j + 2m, -m)$
R2	$y = -m$	$x = j + 2m - 1$	$(j + 2m - 1, -m)$
R3	$x = j + 2m - 1$	$y = m$	$(j + (2m - 1), m)$

At this point there are at most $2m - 1$ distinct components in the link, namely $T_{\frac{n}{2}-(m-1)}, T_{\frac{n}{2}-(m-2)}, \dots, T_{\frac{n}{2}}, T_{\frac{n}{2}+1}, \dots, T_{\frac{n}{2}+m-1}$. We will show that $T_{\frac{n}{2}-i} = T_{\frac{n}{2}+i}$, so that $T_{\frac{n}{2}+i}$ for $i = 0, 1, \dots, m-1$ represent the m components of the link in H. In order to prove this we will use the symmetry of H to trace strands of the components $T_{\frac{n}{2}-i}$ and $T_{\frac{n}{2}+i}$ that connect via a horizontal line in H. So now let $j = \frac{n}{2} + i$, where $i = -(m-1), -(m-2), \dots, m-1$. Note that $\frac{n}{2} - i = n - j$, so we will show that $T_{n-j} = T_j$.

We will now investigate the end behavior of T_j in H, and how it hits the left or right side of H. That is, for different values j, how will the component T_j hit the lower left, upper left, upper right, or lower right edges of H. Given the point (j, m) on T_j, use one of the two cases:

(a) If $1 \le j \le m - 1$ then use the steps to trace the T_j component to the lower left (LL) side of H. As seen below, $y = -j$ will be contained in the T_j component.

Step	Follow along line	Until intersects line	Intersection point
LL1	$x = j$	$x + y = 0$	$(j, -j)$

Since $1 \le j \le m - 1$, then $n - (m - 1) \le n - j \le n - 1$. Starting at the point $(n - j, m)$, use the steps to trace the T_{n-j} component to the lower right (LR) side of H. As seen below, $y = -j$ will be contained in the T_{n-j} component.

Step	Follow along line	Until intersects line	Intersection point
LR1	$x + y = n - j + m$	$x = n + m$	$(n + m, -j)$

Hence T_j and T_{n-j} are connected via the line $y = -j$, and are therefore the same component.

(b) If $m \le j \le 2m - 1$ then use the steps to trace the T_j component to the upper left (LU) side of H. As seen below, $y = j - m$ will be contained in the T_j component.

Step	Follow along line	Until intersects line	Intersection point
UL1	$x = j$	$y = -m$	$(j, -m)$
UL2	$y = -m$	$x = j + 1$	$(j + 1, -m)$
UL3	$x + y = j - m + 1$	$x = 0$	$(0, j - m + 1)$
UL4	$x = 0$	$y = j - m$	$(0, j - m)$

Since $m \le j \le 2m - 1$, then $n - (2m - 1) \le n - j \le n - m$. Starting at the point $(n - j, m)$, use the steps to trace the T_{n-j} component to the upper right (RU) side of H. As seen below, $y = j - m$ will be contained in the T_{n-j} component.

Step	Follow along line	Until intersects line	Intersection point
UR1	$x + y = n - j + m$	$y = -m$	$(n - j + 2m, -m)$
UR2	$y = -m$	$x = n - j + 2m - 1$	$(n - j + 2m - 1, -m)$
UR3	$x = n - j + 2m - 1$	$x + y = n + m$	$(n - j + (2m - 1), j - m - 1))$
UR4	$x + y = n + m$	$y = j - m - 1 + 1$	$(n - j + (2m - 1) + 1, j - m$

Hence T_j and T_{n-j} are connected via the line $y = j - m$, and are therefore the same component.

In each case, we have that $T_{\frac{n}{2}+i} = T_j = T_{n-j} = T_{\frac{n}{2}-i}$, therefore the link in H has at most m components, namely $T_{\frac{n}{2}+i}$ for $i = 0, 1, \ldots, m-1$. We will see that these components have up to three general descriptions given in the cases below. The appearance of these cases depends on the relationship between n and m. In particular, one of the following relations will hold:

- $n \geq 2m$ (H is short and wide)
- $m \leq n < 2m$ (H is somewhat proportional)
- $n \leq m - 1$ (H is tall and thin)

Suppose $n \geq 2m$, then we will only use Case 1 below. Indeed, in Case 1 we have $i = 0, \ldots, \min\{m - 1, \frac{n}{2} - 1\} = m - 1$, where the last equality holds since $n \geq 2m$. We will also have $\frac{n}{2} > m - \frac{n}{2} - 1$, so there are no values of i that fall under Case 2 when $n \geq 2m$. Similarly, if $n \geq 2m$, then $\max\{\frac{n}{2}, m - \frac{n}{2}\} = \frac{n}{2} \geq m > m - 1$. Therefore there are no values of i that fall under Case 3 when $n \geq 2m$. Hence, when $n \geq 2m$, all of our values of i for $i = 0, 1, \ldots, m - 1$ fall under Case 1.

Now suppose $m \leq n < 2m$, then we will use cases 1 and 3 below. Indeed, in Case 1 we have $i = 0, \ldots, \min\{m-1, \frac{n}{2} - 1\} = \frac{n}{2} - 1$, where the last equality holds since $n < 2m$. We will also have $\frac{n}{2} > m - \frac{n}{2} - 1$ since $n > m - 1$. Therefore there are no values of i that fall under Case 2 when $m \leq n < 2m$. On the other hand, since $m \leq n$, then $\max\{\frac{n}{2}, m - \frac{n}{2}\} = \frac{n}{2}$. Therefore $i = \frac{n}{2}, \ldots, m - 1$ fall under Case 3.

Lastly, suppose $n \leq m - 1$, then we will use all three cases below. Indeed, in Case 1 we have $i = 0, \ldots, \min\{m - 1, \frac{n}{2} - 1\} = \frac{n}{2} - 1$, where the last equality holds since $\frac{n}{2} - 1 \leq n \leq m - 1$. Since $n \leq m - 1$, then $\frac{n}{2} \leq m - \frac{n}{2} - 1$. Therefore $i = \frac{n}{2}, \ldots, m - \frac{n}{2} - 1$ fall under Case 2. Lastly, since $n \leq m - 1$, then $\max\{\frac{n}{2}, m - \frac{n}{2}\} = m - \frac{n}{2}$. Therefore $i = m - \frac{n}{2}, \ldots, m - 1$ falls under Case 3.

In all three relationships between n and m, for $i = 0, 1, \ldots, m - 1$, i is accounted for in exactly one of the three cases below.

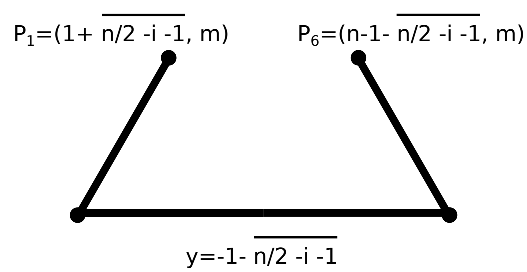
Case 1: For $i = 0, \ldots, \min\{m - 1, \frac{n}{2} - 1\}$, the components $T_{\frac{n}{2}-i}$, and $T_{\frac{n}{2}+i}$ can be described as follows. The last points in the top left of H bouncing to the left from $(\frac{n}{2} - i, m)$ and $(\frac{n}{2} + i, m)$ respectively are $(1 + \overline{\frac{n}{2} - i - 1}, m)$ and $(1 + \overline{\frac{n}{2} + i - 1}, m)$. Using the symmetry of H, the last points on the top right of H bouncing from the right $(\frac{n}{2} - i, m)$ and $(\frac{n}{2} + i, m)$ respectively are $(n - 1 - \overline{\frac{n}{2} + i - 1}, m)$ and $(n - 1 - \overline{\frac{n}{2} - i - 1}, m)$.

Notice that a strand of the component is created from tracing $T_{\frac{n}{2}-i}$ to the left and $T_{\frac{n}{2}+i}$ to the right; this is given by (i) or (ii) below. Likewise a strand of the component is created from tracing $T_{\frac{n}{2}-i}$ to the right and $T_{\frac{n}{2}+i}$ to the left; this is given by (iii) and (iv) cases below. Let

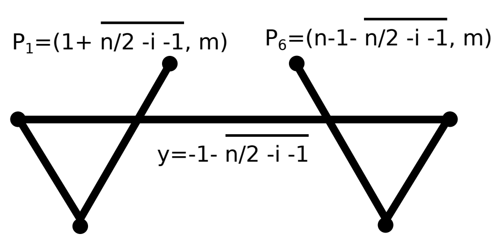
$$\begin{aligned}
P_1 &= (1+\overline{\frac{n}{2}-i-1}, m),\\
P_2 &= (1+\overline{\frac{n}{2}+i-1}, m),\\
P_3 &= (\frac{n}{2}-i, m),\\
P_4 &= (\frac{n}{2}+i, m),\\
P_5 &= (n-1-\overline{\frac{n}{2}+i-1}, m),\\
P_6 &= (n-1-\overline{\frac{n}{2}-i-1}, m).
\end{aligned}$$

Note that P_1, P_2, P_3, are symmetric (across the point $(\frac{n}{2}, m)$) to P_6, P_5, P_4 respectively.

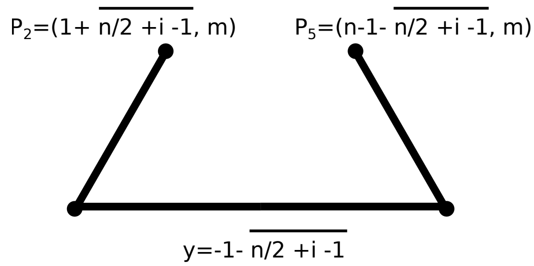
(i) If $1 \leq 1+\overline{\frac{n}{2}-i-1} \leq m-1$, then apply part (a) to the point P_1. We see that $T_{1+\overline{\frac{n}{2}-i-1}} = T_{\frac{n}{2}-i}$ is connected to $T_{n-(1+\overline{\frac{n}{2}-i-1})} = T_{\frac{n}{2}+i}$ (and therefore P_1 is connected to P_6) via the line $y = -1-\overline{\frac{n}{2}-i-1}$. In this case, this segment of $T_{\frac{n}{2}+i}$ will have the form below.



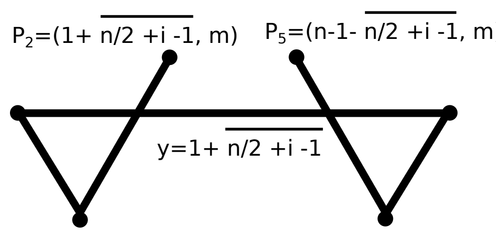
(ii) If $m \leq 1+\overline{\frac{n}{2}-i-1} \leq 2m-1$, then apply part (b) to the point P_1. We see that $T_{1+\overline{\frac{n}{2}-i-1}} = T_{\frac{n}{2}-i}$ is connected to $T_{n-(1+\overline{\frac{n}{2}-i-1})} = T_{\frac{n}{2}+i}$ (and therefore P_1 is connected to P_6) via the line $y = 1+\overline{\frac{n}{2}-i-1}-m$. In this case, this segment of $T_{\frac{n}{2}+i}$ will have the form below.



(iii) If $1 \leq 1+\overline{\frac{n}{2}+i-1} \leq m-1$, then apply part (a) to the point P_2. We see that $T_{1+\overline{\frac{n}{2}+i-1}} = T_{\frac{n}{2}+i}$ is connected to $T_{n-(1+\overline{\frac{n}{2}+i-1})} = T_{\frac{n}{2}-i}$ (and therefore P_2 is connected to P_5) via the line $y = -1 - \overline{\frac{n}{2} + i - 1}$. In this case, this segment of $T_{\frac{n}{2}+i}$ will have the form below.



(iv) If $m \leq 1 + \overline{\frac{n}{2} + i - 1} \leq 2m - 1$, then apply part (b) to the point P_2. We see that $T_{1+\overline{\frac{n}{2}+i-1}} = T_{\frac{n}{2}+i}$ is connected to $T_{n-(1+\overline{\frac{n}{2}+i-1})} = T_{\frac{n}{2}-i}$ (and therefore P_2 is connected to P_5) via the line $y = 1 + \overline{\frac{n}{2} + i - 1} - m$. In this case, this segment of$T_{\frac{n}{2}+i}$ will have the form below.



Notation. We write $P_k \to P_l$ if we can travel along a component T to get from the point P_k to the point P_l. Note that $P_k \to P_l$ if and only if $P_l \to P_k$.

Cases (i) and (ii) show that $P_1 \to P_6$, while cases (iii) and (iv) show that $P_2 \to P_5$. Moreover, we know that $P_1 \to P_3 \to P_5$ since P_1 and P_3 are a factor of $(2m-1)$ away from each other, and P_3 and P_5 are a factor of $(2m-1)$ away from each other. We will similarly have $P_2 \to P_4 \to P_6$. Therefore, if we follow $T_{\frac{n}{2}-i} = T_{\frac{n}{2}+i}$ starting from P_1, then we have

$$P_1 \to P_3 \to P_5 \to P_2 \to P_4 \to P_6 \to P_1,$$

thus completing the component.

Observation: The P_k's are not necessarily distinct.

Case 2: For $i = \frac{n}{2}, \ldots, m - \frac{n}{2} - 1$, the components $T_{\frac{n}{2}-i}$, and $T_{\frac{n}{2}+i}$ can be described in the following way. Since $1 \leq \frac{n}{2} + i \leq m - 1$, apply (a) to the point $(\frac{n}{2} + i, m) = (j, m)$. From part (a), we see that $T_{\frac{n}{2}+i}$ and $T_{\frac{n}{2}-i}$ are connected via the line $y = -\frac{n}{2} - i$. Note that for $i > \frac{n}{2}$, the points $(\frac{n}{2} - i, m)$ and $(\frac{n}{2} + i, m)$ are not in H. However, we use the points $(0, \frac{n}{2} - i + m)$ and $(\frac{n}{2} + i, m + \frac{n}{2} - i)$ on $T_{\frac{n}{2}-i}$, and $T_{\frac{n}{2}+i}$ respectively. See that the line $y = \frac{n}{2} - i + m$ connects these new points, and the component $T_{\frac{n}{2}+i} = T_{\frac{n}{2}-i}$ is complete.

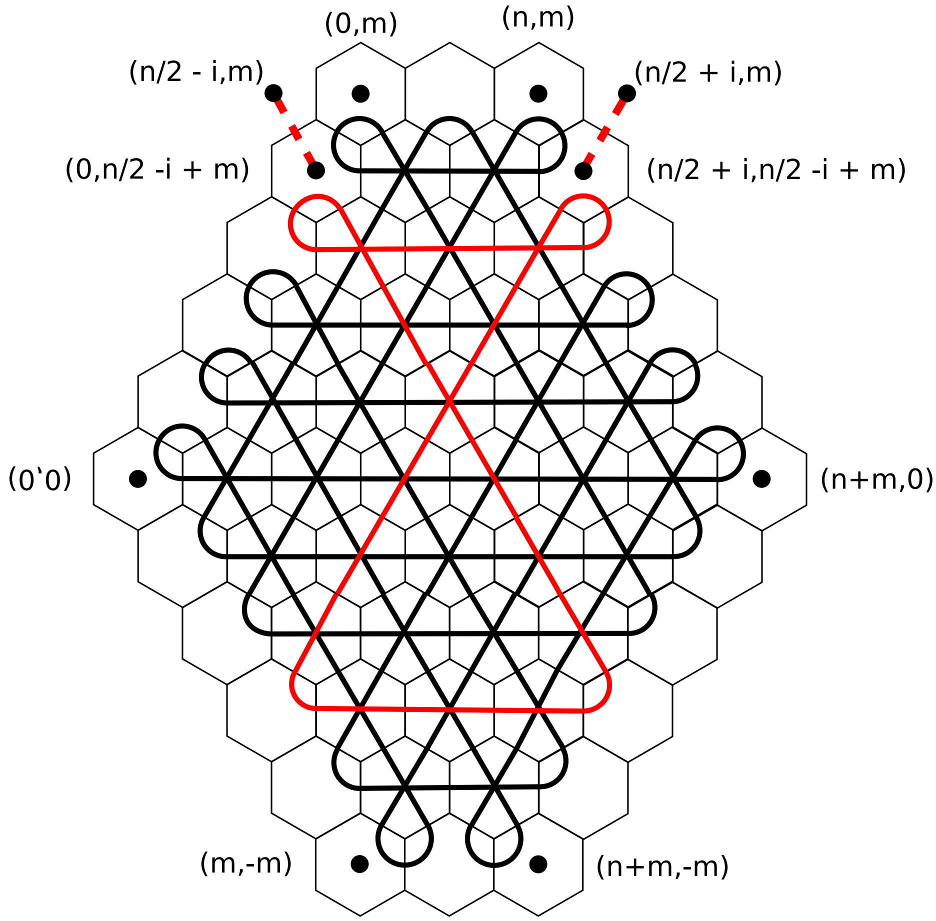


Figure 33: Case 2 with $i = 2$

Case 3: For $i = \max\{\frac{n}{2}, m - \frac{n}{2}\}, \ldots, m-1$, the components $= T_{\frac{n}{2}-i}$, and $T_{\frac{n}{2}+i}$ can be described in the following way. Note that this case only exists if $\frac{n}{2} \leq m-1$. Therefore

$$m \leq \max\{n, m\} \leq \frac{n}{2} + i \leq \frac{n}{2} + m - 1 \leq 2(m-1) \leq 2m - 1$$

so we can apply (b) to the point $(\frac{n}{2}+i, m) = (j, m)$. From part (b), we know that $T_{\frac{n}{2}-i}$ and $T_{\frac{n}{2}+i}$ are connected via the line $y = \frac{n}{2} + i - m$. Note that for $i > \max\{\frac{n}{2}, m - \frac{n}{2}\}$, the points $(\frac{n}{2} - i, m)$ and $(\frac{n}{2} + i, m)$ are not in H. However, as in Case 2, we use the points $(0, \frac{n}{2} - i + m)$ and $(\frac{n}{2} + i, m + \frac{n}{2} - i)$ on $T_{\frac{n}{2}-i}$ and $T_{\frac{n}{2}+i}$ respectively. See that the line $y = \frac{n}{2} - i + m$ connects these new points, and the component $T_{\frac{n}{2}+i} = T_{\frac{n}{2}-i}$ is complete.

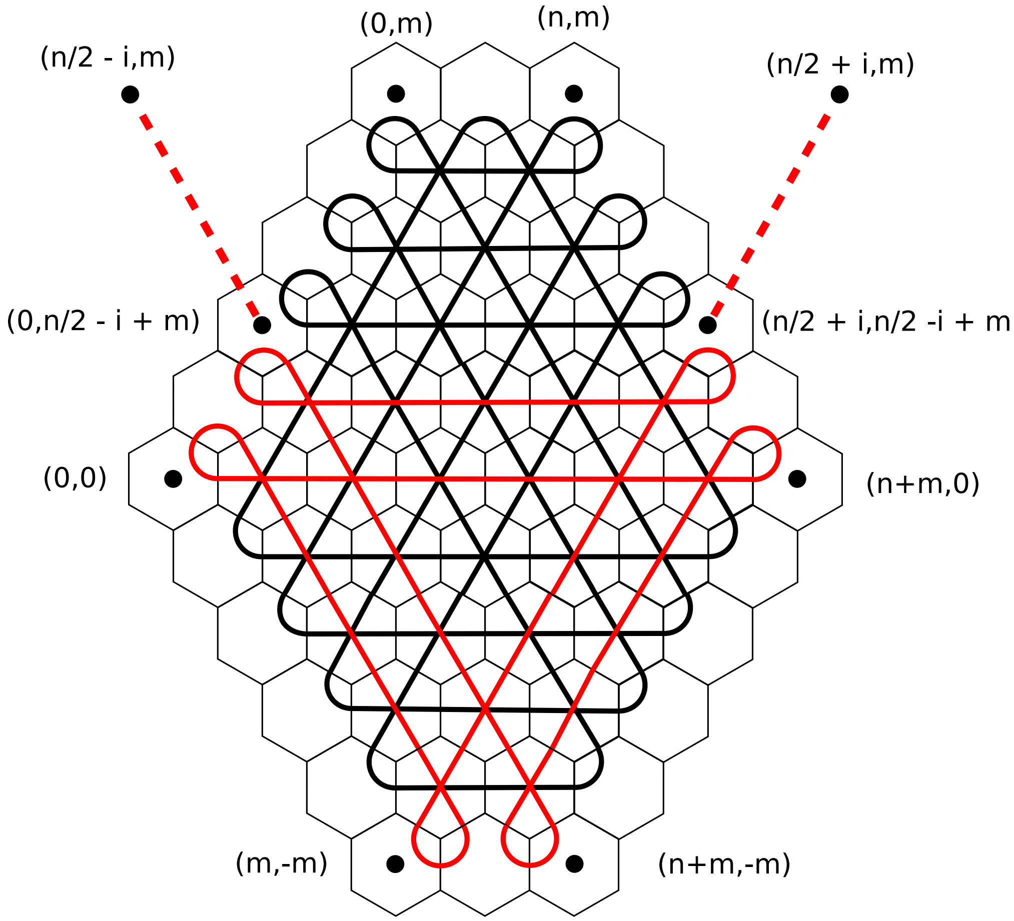


Figure 34: Case 3 with $i = 4$

Now that we have described the components of the link created by H, let us focus our attention to counting them. We will show that H has exactly m components by keeping track of how many times each component uses a horizontal lines across H. In Case 1, Case 2, and Case 3, each component uses at most two horizontal lines across H. I claim that there exists a unique $i^* \in \{\frac{n}{2}, \frac{n}{2}+1, \ldots, \frac{n}{2}+m-1\}$ such that $T_{\frac{n}{2}+i^*}$ uses exactly one horizontal line and $T_{\frac{n}{2}+i}$ uses two distinct horizontal lines for $i \neq i^*$.

Take any $i \in \{1, 2, \ldots, m-1\}$. Suppose i falls under Case 2. Then, $T_{\frac{n}{2}+i}$ uses the horizontal lines $y_1 = -\frac{n}{2} - i$ and $y_2 = \frac{n}{2} - i + m$. Hence $y_1 = y_2$ if and only if $n = -m$. However $n \geq 0$ so we have reached a contradiction. So if i falls under Case 2, then $T_{\frac{n}{2}+i}$ uses two distinct horizontal lines.

Now suppose i falls under Case 3. Then $T_{\frac{n}{2}+i}$ uses the horizontal lines $y_1 = \frac{n}{2} + i - m$ and $y_2 = \frac{n}{2} - i + m$. Therefore $y_1 = y_2$ if and only if $i = m$. However, for Case 3, we know that $i \leq m - 1 < m$, so we have reached a contradiction. Therefore if i falls under Case 3, then $T_{\frac{n}{2}+i}$ uses two distinct horizontal lines.

Finally, suppose i falls under Case 1. Then $T_{\frac{n}{2}+i}$ uses the lines

$$y_1 = -1 - \overline{\frac{n}{2} - i - 1} \qquad \text{or} \qquad y_2 = 1 + \overline{\frac{n}{2} - i - 1} - m$$

and

$$y_3 = -1 - \overline{\frac{n}{2} + i - 1} \qquad \text{or} \qquad y_4 = 1 + \overline{\frac{n}{2} + i - 1} - m.$$

Note that $y_1 < 0, y_2 \geq 0, y_3 < 0$, and $y_4 \geq 0$. Therefore $y_1 \neq y_2, y_1 \neq y_4, y_2 \neq y_3$, and $y_3 \neq y_4$. Hence we only need to check for what values of i does $y_1 = y_3$ and $y_2 = y_4$. We have $y_1 = y_3$ if and only if

$$-1 - \overline{\frac{n}{2} - i - 1} = -1 - \overline{\frac{n}{2} + i - 1}$$

if and only if

$$\overline{0} = \overline{2i}.$$

This implies that either $\overline{i} = \overline{0}$ or i has order 2 in the group $\langle \mathbb{Z}/(2m-1)\mathbb{Z}, + \rangle$. However, $2 \nmid (2m-1)$, therefore $|i| \neq 2$, and so $\overline{i} = \overline{0}$. Hence $y_1 = y_3$ if and only if $\overline{i} = \overline{0}$. We will similarly have $T_2 = T_4$ if and only if $\overline{0} = \overline{2i}$ if and only if $\overline{i} = \overline{0}$. In Case 1 we have $0 \leq i \leq \min\{m-1, \frac{n}{2} - 1\}$. Therefore $i^* = 0$ is the only value such that $\overline{i^*} = \overline{0}$ and is therefore the only value such that $T_{\frac{n}{2}+i^*}$ uses exactly one horizontal line.

Every horizontal line $-(m-1) \leq y \leq m-1$ appears in one and only one component. From the work above, $T_{\frac{n}{2}}$ used one horizontal line and $T_{\frac{n}{2}+1}, \ldots, T_{\frac{n}{2}+(m-1)}$ used two horizontal lines each. This captures all horizontal lines to be used in H. Hence $T_{\frac{n}{2}}, T_{\frac{n}{2}+1}, \ldots, T_{\frac{n}{2}+(m-1)}$ are the distinct m components of the link created by H.

2. Suppose n is odd. Consider points of the form $(j, -(m-1))$, rather than (j, m) as in the n even case, and let B_j be the component of the link produced by H containing segments of the lines $x = j$ or $x + y = j - (m-1)$. Just as $T_j = T_k$ if $\overline{j} = \overline{k}$, where $\overline{j}$ denotes $j \mod (2m-1)$, we can show that $B_j = B_k$ if $\overline{j} = \overline{k}$. Therefore we again have at most $2m - 1$ distinct components in the link produced by H, namely

$$B_{m+\frac{n-1}{2}-(m-1)}, B_{m+\frac{n-1}{2}-(m-2)}, \ldots, B_{m+\frac{n-1}{2}}, B_{m+\frac{n-1}{2}+1}, \ldots, B_{m+\frac{n-1}{2}+m-1}.$$

As in the n even case, there is symmetry among H. In particular, we will have $B_{m+\frac{n-1}{2}+i} = B_{m+\frac{n-1}{2}-i}$. An example of this can be seen in red for $i = 1$ in Figure **??** below. This implies that the link in H has at most m components, namely $B_{m+\frac{n-1}{2}+i}$ for $i = 0, 1, \ldots, m-1$. Using similar arguments to the n even case, we get that $B_{m+\frac{n-1}{2}}$ uses exactly one horizontal line and $B_{m+\frac{n-1}{2}+1}, B_{m+\frac{n-1}{2}+2}, \ldots, B_{m+\frac{n-1}{2}+m-1}$ uses two horizontal lines each, which we know implies that H produces a link with exactly m components.

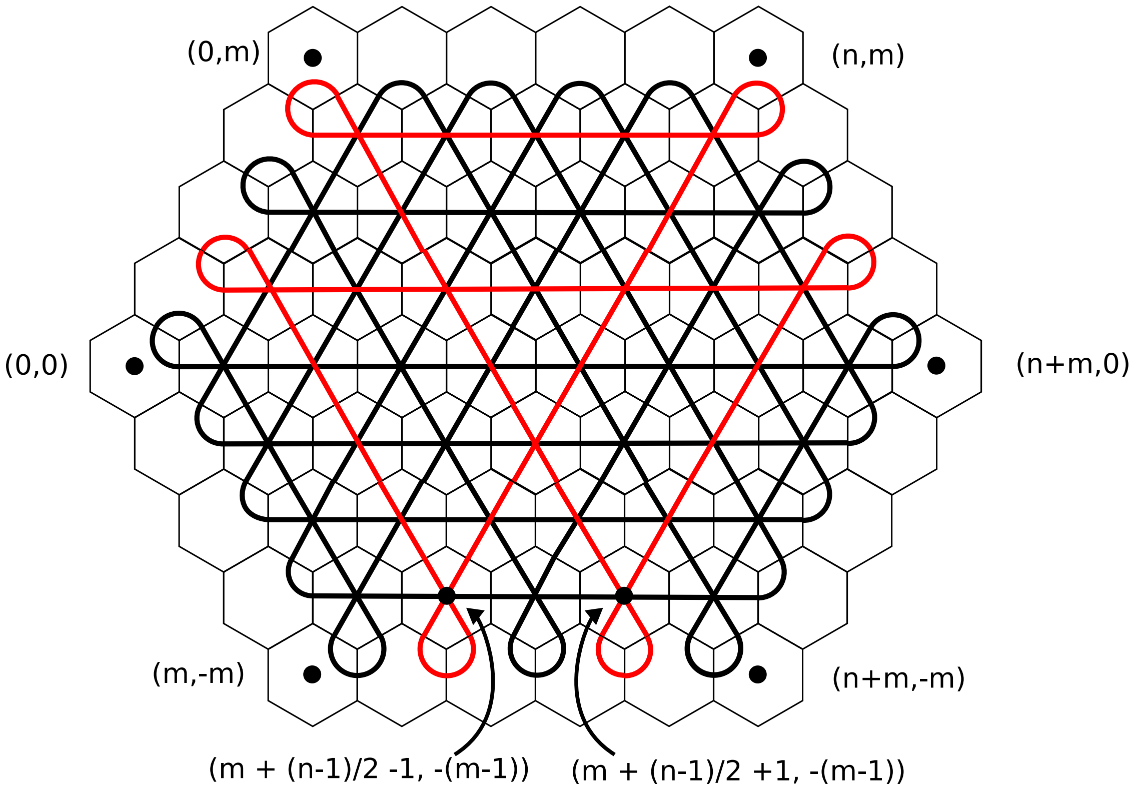


Figure 35: A 4×5 saturated stretched hexagon.

□

In the figures below, we have colored each component a distinct color. For both figures we indeed have m components.

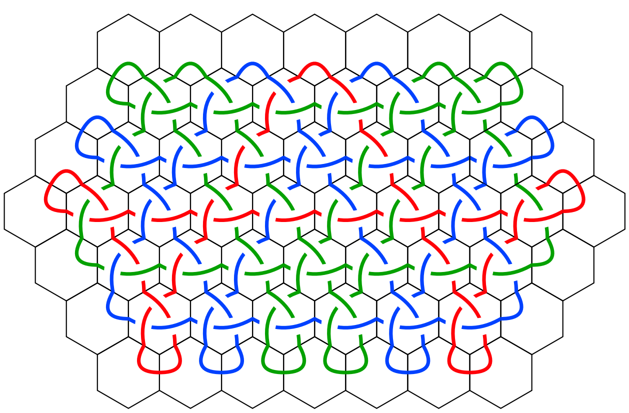


Figure 36: A 3×6 saturated stretched hexagon.

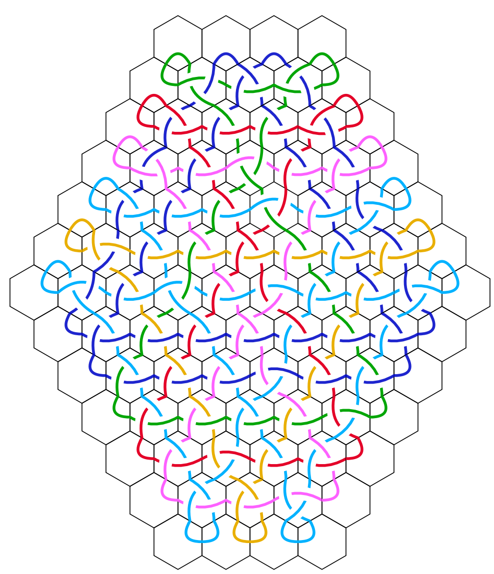


Figure 37: A 6×3 saturated stretched hexagon

4 Future Work

We end with a few open questions to consider in the future.

1) Does there exist a family of knots whose corona number is realized only when its crossing number is not?

   In the square mosaic case, Evans, Ludwig, and Paat were able to show that there exists a family of knots whose mosaic number is realized only when their crossing number is not [3]. Can we doe the same for hexagonal tiles?

2) Can we find an upper bound for the corona number for a specific family of knots?

   For example, if K is the composition of n trefoils, does there exist an upper bound for the corona number of K? What if K is a (p, q) torus knot?

3) Is there a way to restrict our set of hextiles to produce a more interesting knot invariant?

   Out of all the knots K whose corona number was established in this paper, all but two of them had corona number equal to 2. Can we restrict our set of hextiles so that the knots with 9 or few crossings have a wider range of values for corona number?

4) What kinds of knots are the m components that are produced by an $m \times n$ saturated stretched hexagon H? Does this differ depending on the relationship between m and n?

   From Theorem **??**, we know H produces a link with m components. If we consider these m components individually, can we establish what kinds of knots the m components are? How does this change as H goes from a short and wide hexagon, to a somewhat proportional hexagon, to a tall and skinny hexagon? Does this depend on where we decide to use T_{20} hextiles versus T_{21} hextiles?

5) Can we measure the "splittability" of the link produced in saturated stretched hexagons?

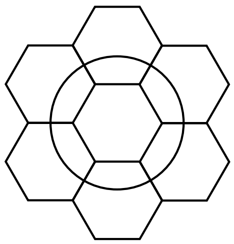
   Note that throughout the proof of Theorem **??**, the crossing tiles in the inner board of H are left ambiguous. Is there a way to suitably connect H with T_{20} and T_{21} to reduce the "splittability" of the link produced in H? A link is *splittable* if the components of the link can be deformed do that they lie on different sides of a plan in three-space [1]

6) What happens when we saturate diagrams of different shapes: hexagons with three sides of different lengths, other irregular polygons?

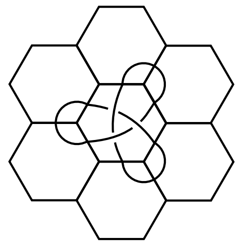
7) What happens when we saturate polygons in a way that allows crossing tiles on the perimeter?

   Recall that our current definition of saturation does not allow crossing tiles on the perimeter. If we allow crossings on the perimeter, will we still get m components? Will H still have symmetric properties?

Appendix



Unknot



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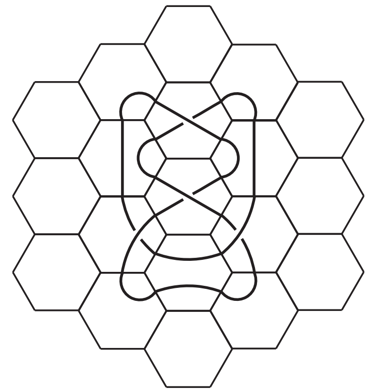

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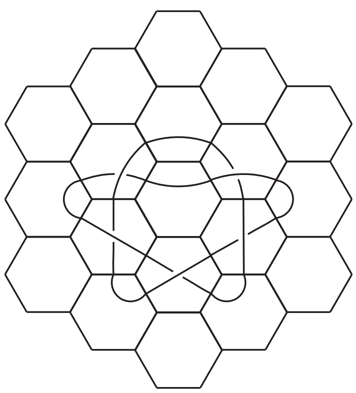

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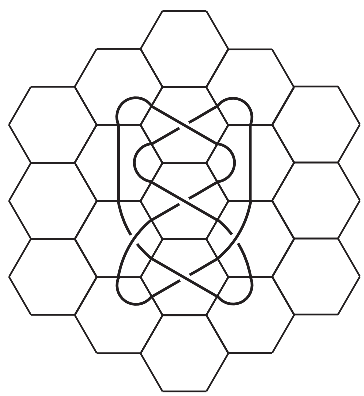

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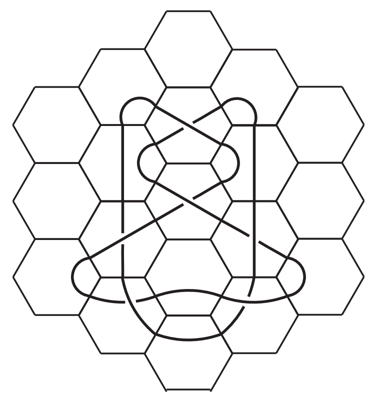

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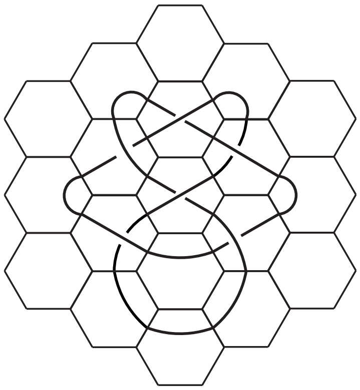

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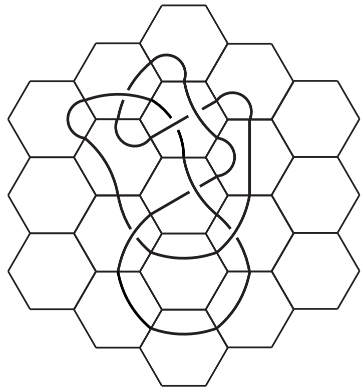

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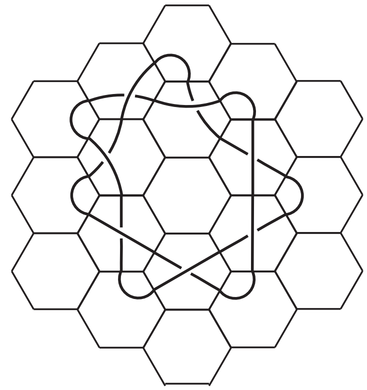

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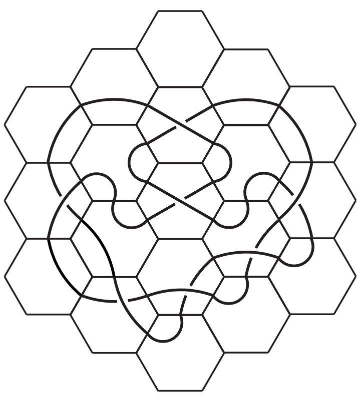

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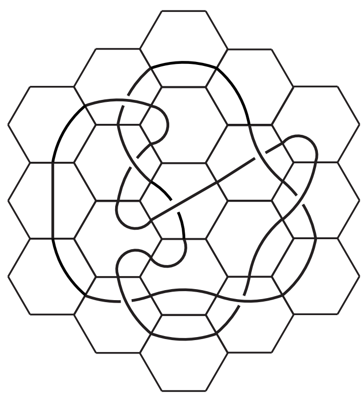

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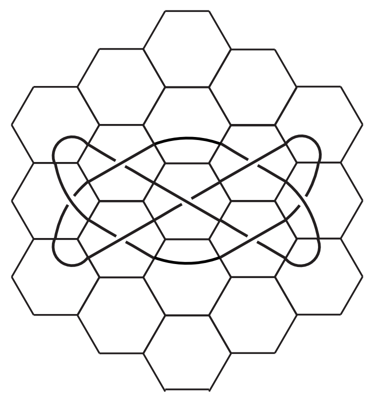

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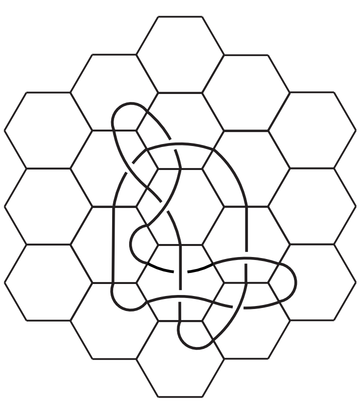

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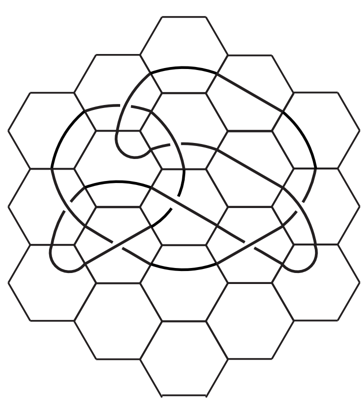

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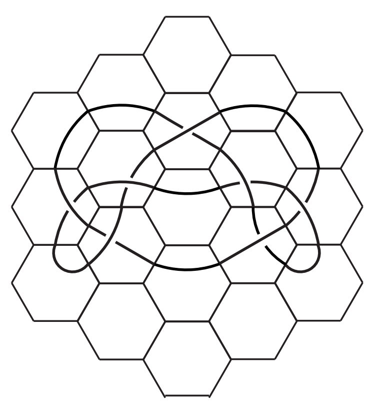

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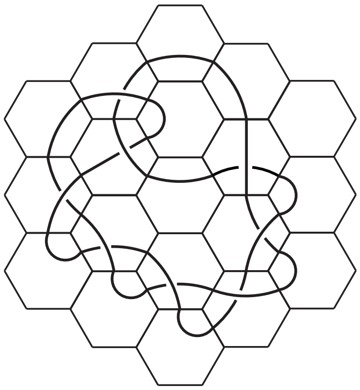

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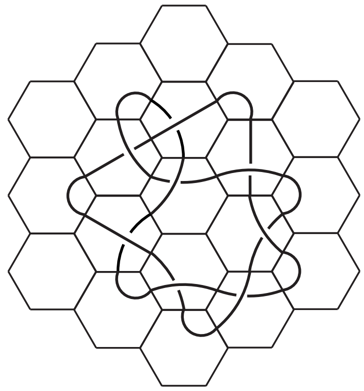

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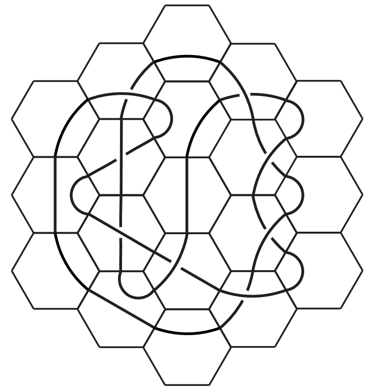

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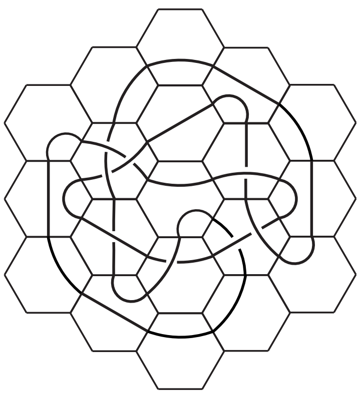

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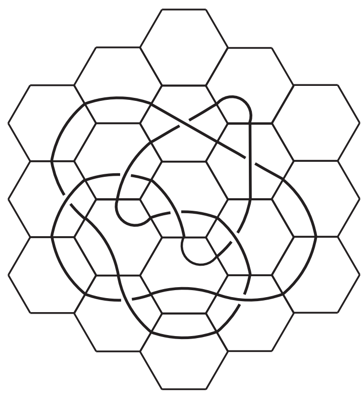

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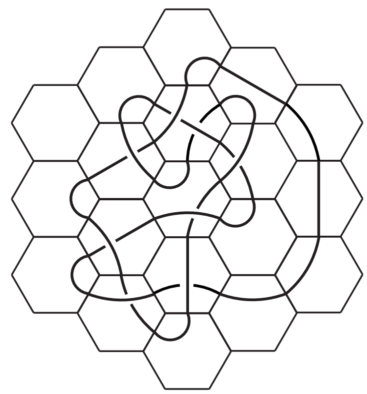

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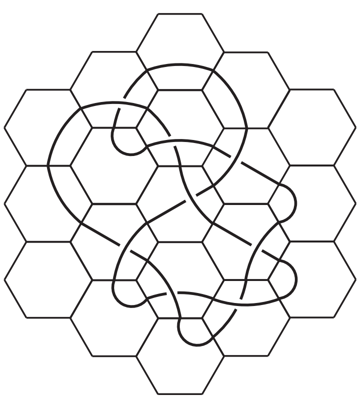

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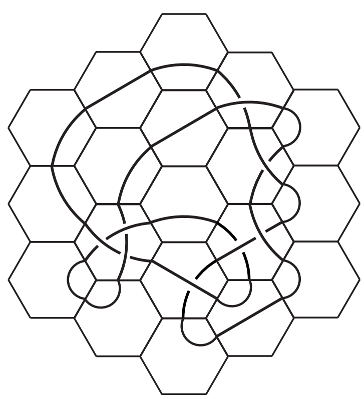

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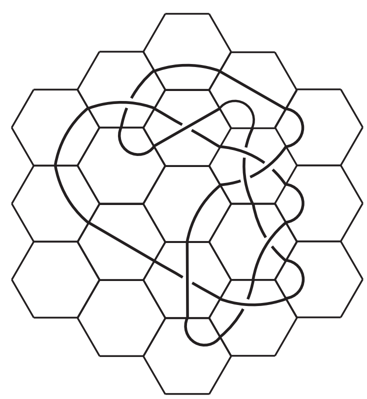

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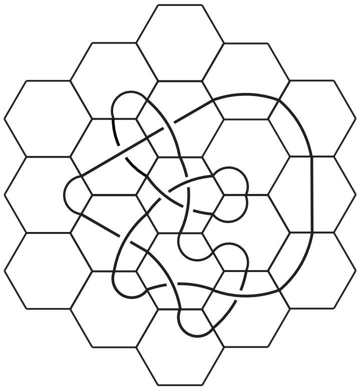

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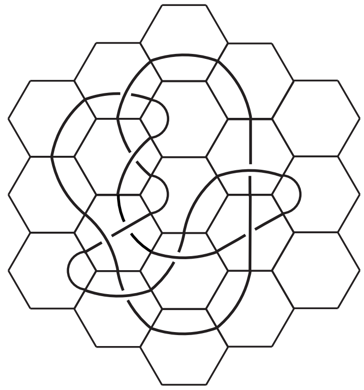

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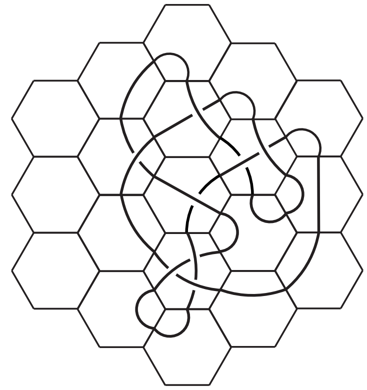

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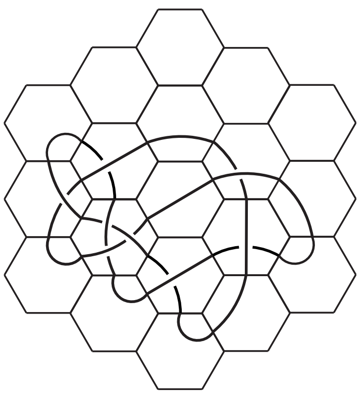

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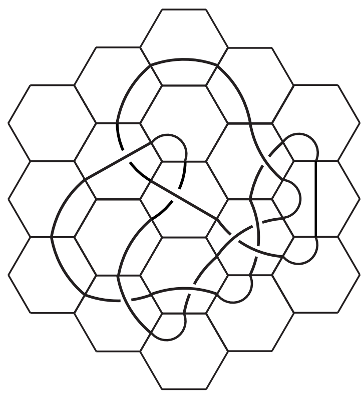

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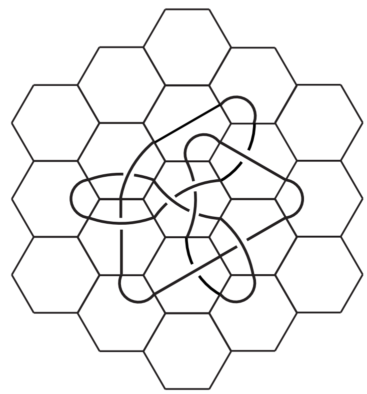

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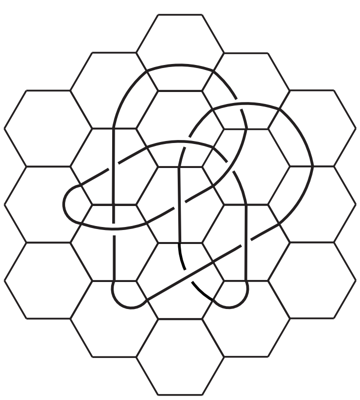

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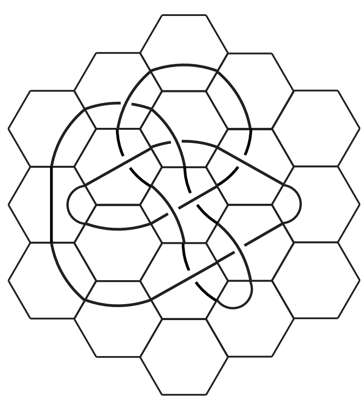

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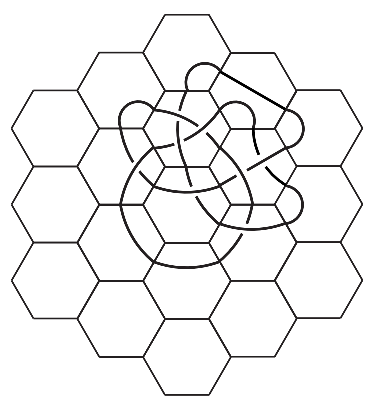

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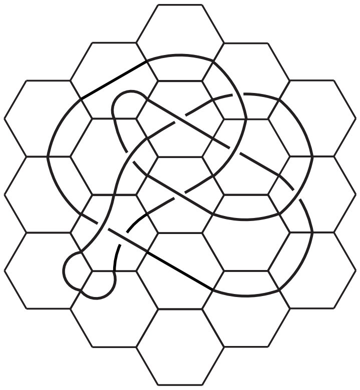

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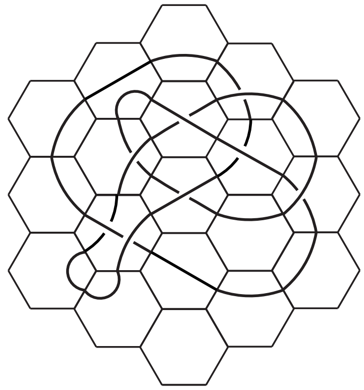

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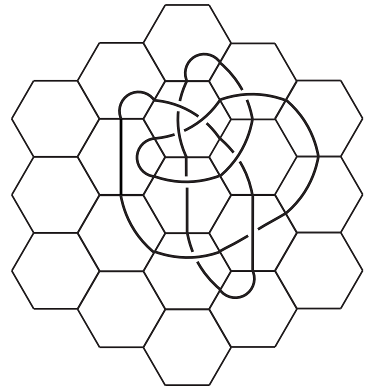

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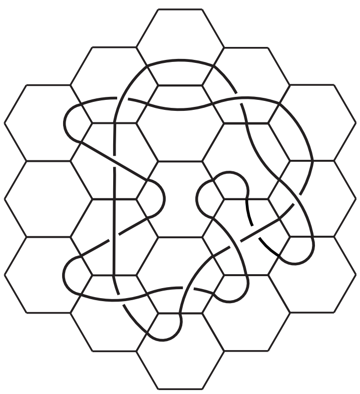

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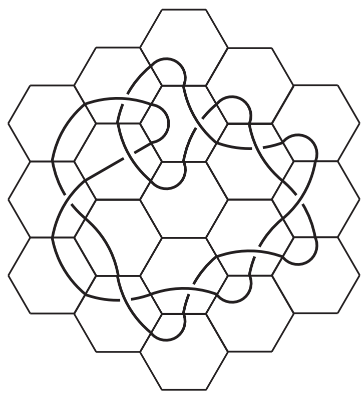

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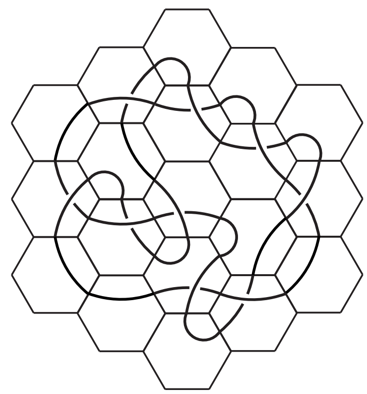

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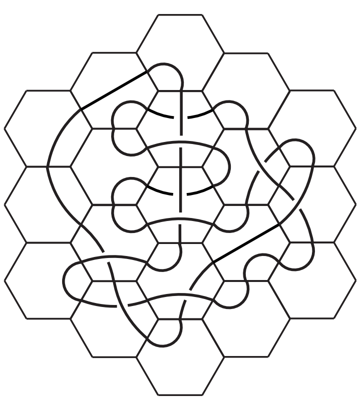

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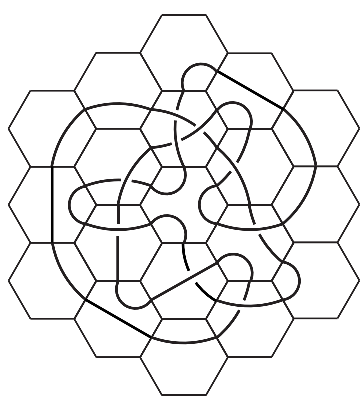

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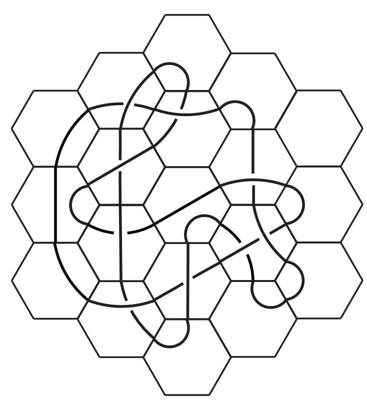

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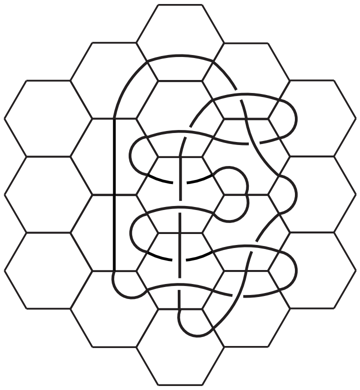

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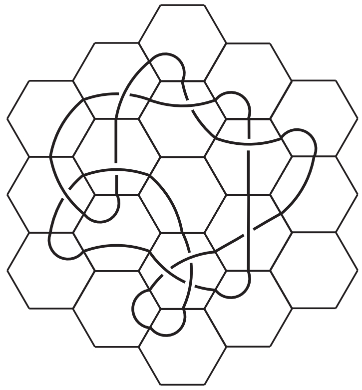

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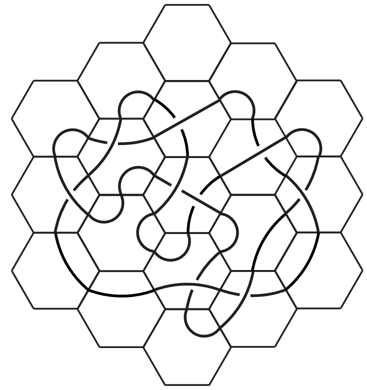

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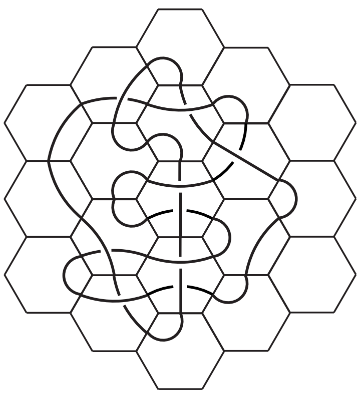

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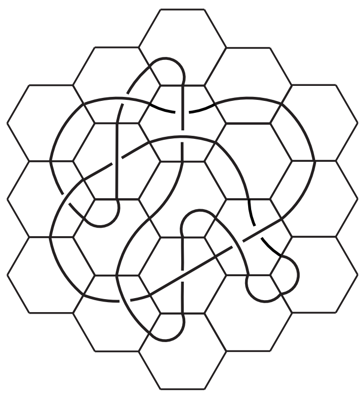

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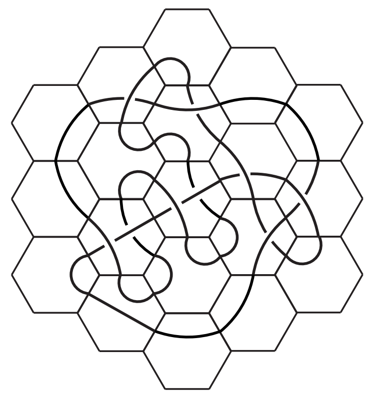

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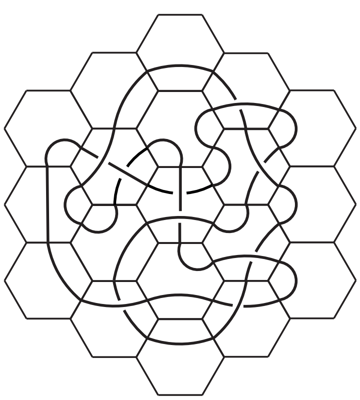

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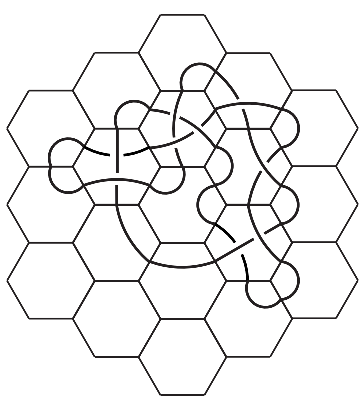

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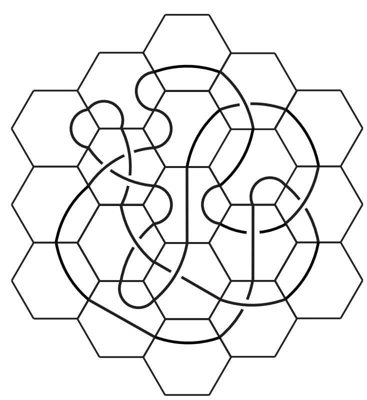

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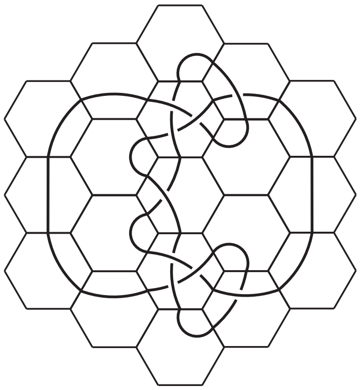

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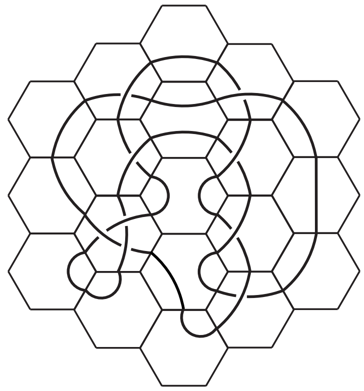

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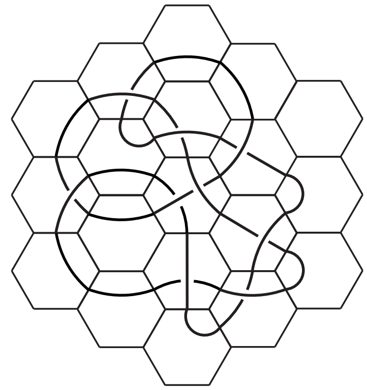

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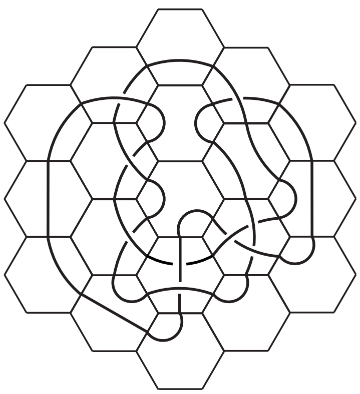

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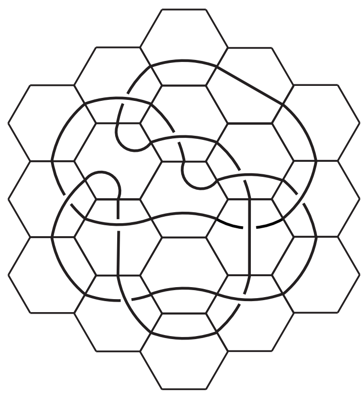

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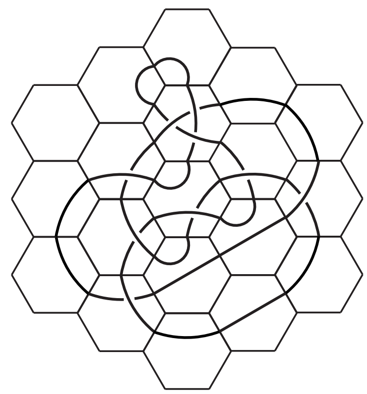

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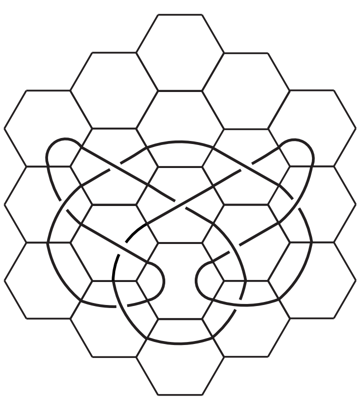

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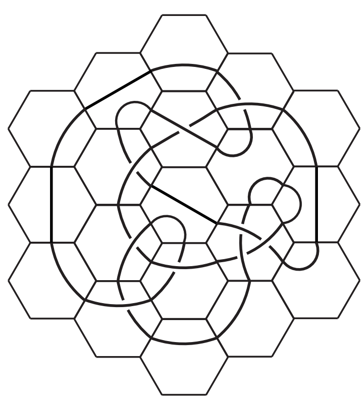

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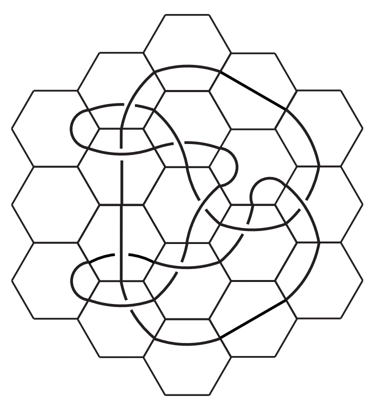

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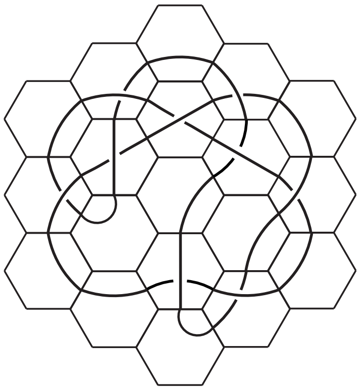

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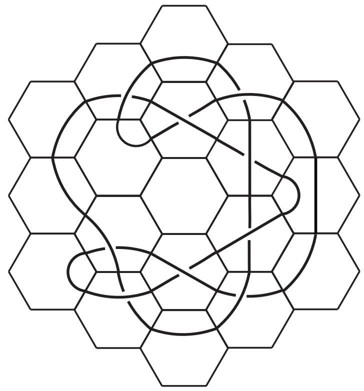

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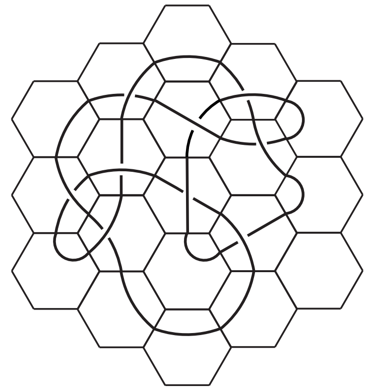

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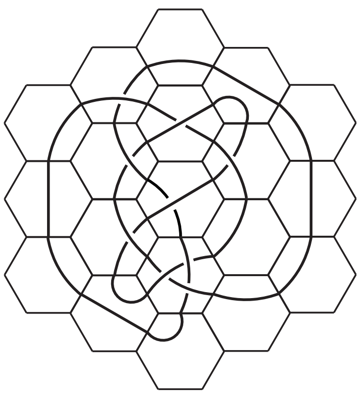

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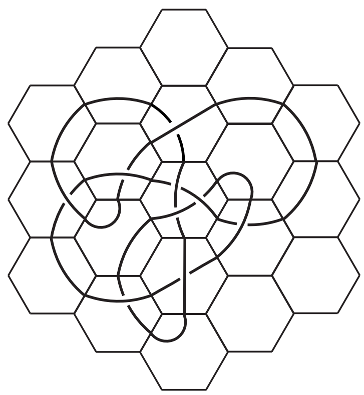

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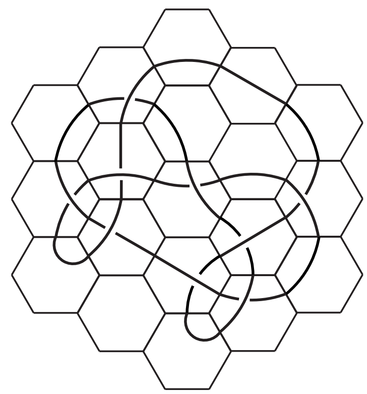

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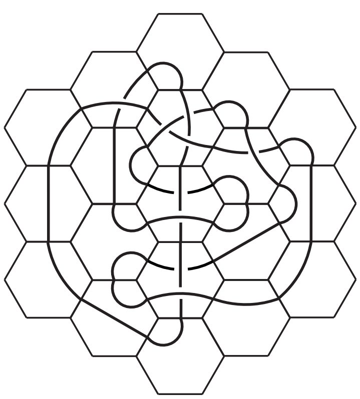

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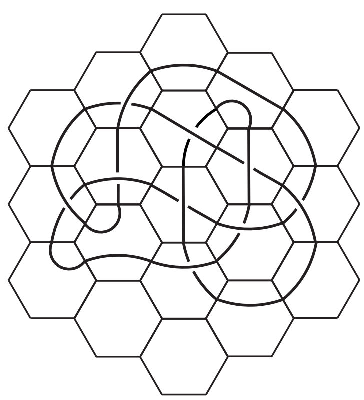

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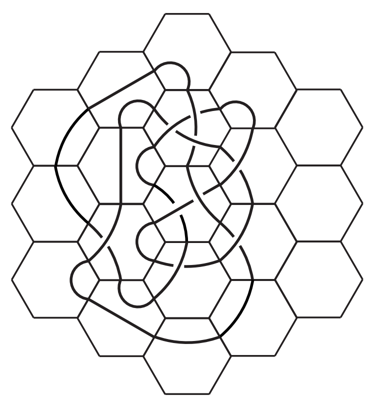

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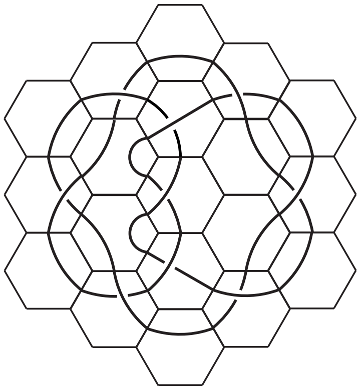

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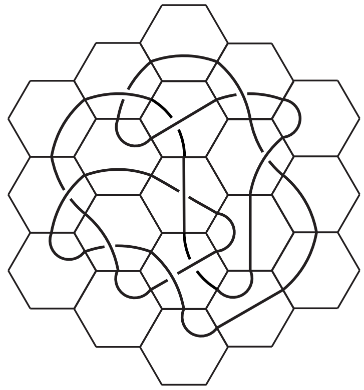

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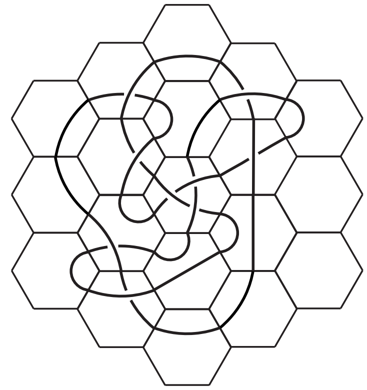

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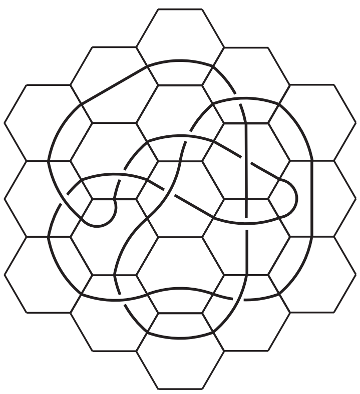

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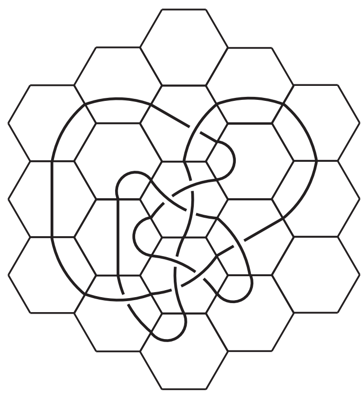

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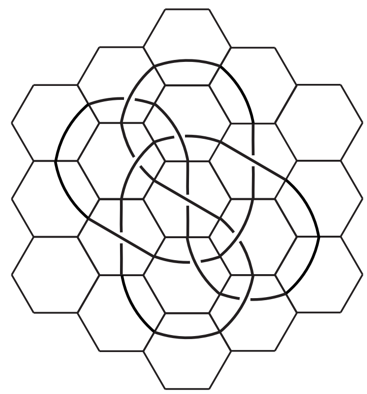

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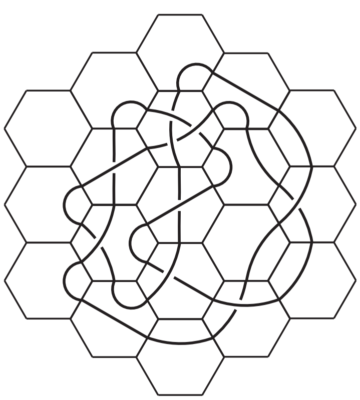

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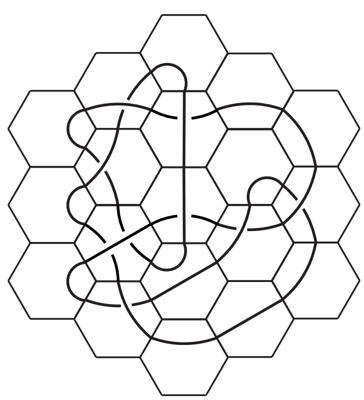

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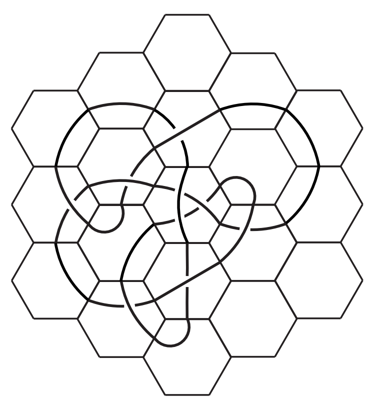

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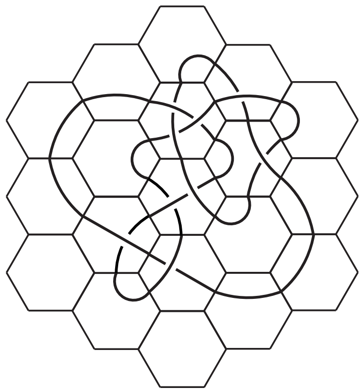

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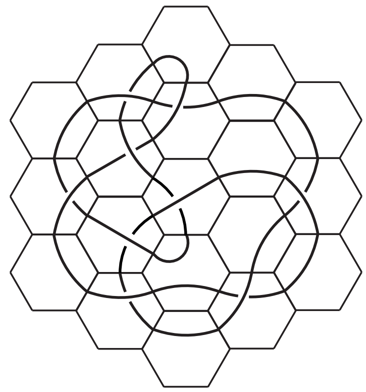

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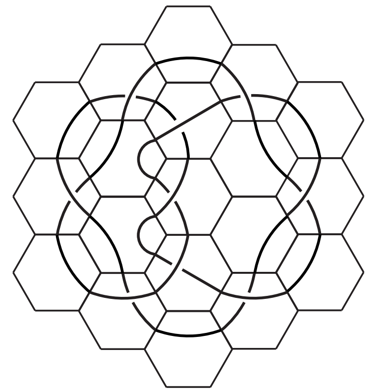

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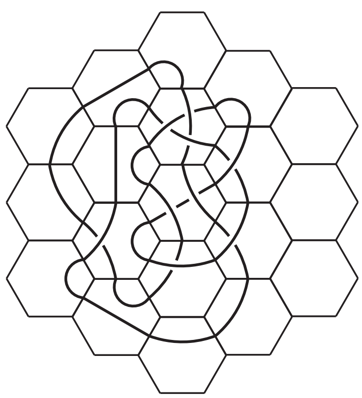

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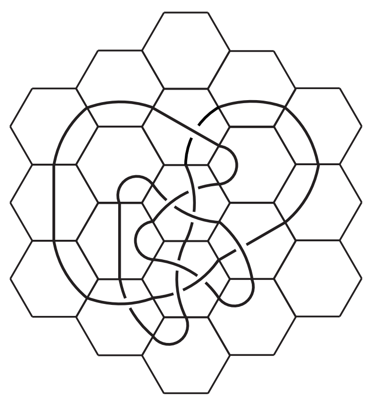

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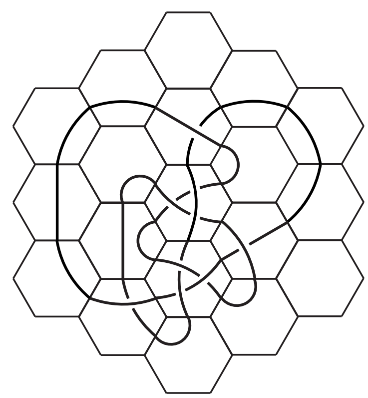

9_{49}

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