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# Computing Symmetric Belyi Maps on Surfaces of High Genus

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Riemann Surfaces can be described either by an algebraic or a complex structure. Belyi's Theorem gives an analytic way to interpret algebraic surfaces i.e. Riemann Surfaces defined by systems of equations with algebraic coefficients over  $\mathbb{Q}$ . They are interpreted as surfaces  $S$  with maps  $f : S \rightarrow \mathbb{P}_{\mathbb{C}}^1$  ramified at most over  $\{0, 1, \infty\}$ . Such a map is called a Belyi Map. A Belyi pair  $(S, f)$  consists of a surface  $S$  and a Belyi Map  $f$  from that surface to  $\mathbb{P}_{\mathbb{C}}^1$ . The absolute Galois Group  $Gal(\overline{\mathbb{Q}}, \mathbb{Q})$  acts on Belyi pairs by acting on the coefficients of the defining equation for the surface  $S$ , and acting on the coefficients of the rational map  $f$ .

With Belyi's Theorem we can learn about algebraic surfaces by studying the combinatorics of bipartite graphs embedded on surfaces. These graphs are realized as the preimage of the interval  $[0, 1]$  under a Belyi Map. Computing Belyi Maps of high degree is difficult, and although significant progress has been made for computations in the plane, there are fewer examples of high degree maps placed on high genus surfaces in the literature.

We will use the symmetries of certain infinite families of graphs in order to place them as dessins on Riemann surfaces of high genus, and compute the corresponding Belyi Maps.

Note that in this paper graphs will always be assumed to be connected. Loops and multiple edges are allowed.

**Definition 0.1.** *A map is a graph  $G$  along with an embedding  $\phi : G \rightarrow S$  of the graph into an oriented 2 dimensional surface  $S$ . Such that:*

- 1 *Edges do not intersect (except at vertices). Equivalently  $\phi$  is injective.*
- 2 *The complement of the embedded graph is a disjoint union of disks.  $(S \setminus \phi(G) = \bigsqcup_{i=1}^n D_i$  where each  $D_i$  is homeomorphic to a disk).*

Each disk  $D_i$  will be called a face of the map  $(G, \phi)$ . The genus of the map  $(G, \phi)$  is just the genus of the surface  $S$ . Once we have an embedded map this genus can be computed as  $V - E + F = 2 - 2g$ .

**Definition 0.2.** *A hypermap is a bipartite map, i.e. a map with its vertices colored black and white, such that adjacent vertices have opposite colors.*

Note that every map can be turned into a hypermap by adding a white vertex in the middle of each edge. If all white vertices are degree 2 we will omit them from drawings and may refer to the hypermap as a map.

**Definition 0.3.** *A segment joining a black and white vertex of a hyper map is called germ of an edge or simply a germ.*

Note that this definition of germ has nothing to do with the germ of a function.

**Definition 0.4.** *The degree of a hypermap is the number of its germs. The degree of a vertex is the number of germs incident with it. The degree of a face is half the number of germs bordering it.*

It will become clear why we use only half the germs bordering a face.

**Definition 0.5.** *The Euler Characteristic  $\chi(M)$  of a hypermap  $M$  on a surface  $S$  is defined to be  $\chi(S)$ .*

Let  $M$  be a hypermap on a surface  $S$ . Let  $B$  be the number of black vertices of  $M$ . Let  $W$  be the number of white vertices of  $M$ . Let  $m$  be the degree of  $M$ . Let

$F$  be the number of faces of  $M$ . Then we can compute the Euler Characteristic  $\chi(M)$ :

$$\chi(M) = 2 - 2g = B + M + F - m = V + F - m.$$

We call a hypermap on the sphere a plane hypermap by convention.

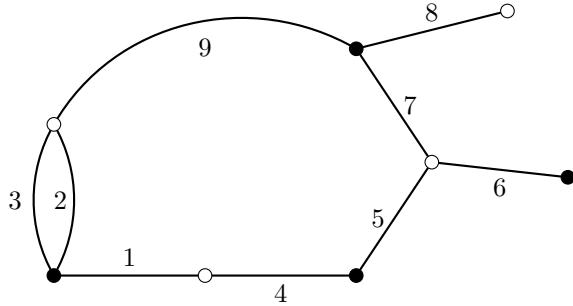
### 1. ENCODING HYPERMAPS BY PERMUTATIONS

We use the fact that the surface is oriented to

- 1 distinguish left and right-hand side when moving along a germ.
- 2 give the surface a positive (counterclockwise) and negative (clockwise) orientation.

Let  $M$  be a hypermap of degree  $m$ . We label the germs from 1 to  $m$  and place the label of each germ near its left side as we travel along a germ from its black end to its white end. Then we associate to  $M$  the triple of permutations  $(\sigma, \alpha, \psi)$ .

- $\sigma$  corresponds to black vertices. Let  $b \in B$  be a black vertex of  $M$ . Then we define  $c(b)$  to be the cycle consisting of the labels on the germs incident to the black vertex  $b$  taken in the positive direction around the vertex. Let the set of black vertices  $B = \{b_1, \dots, b_n\}$ , then  $\sigma = c(b_1) \cdot c(b_2) \cdot \dots \cdot c(b_n)$  is the product of the permutations corresponding to each black vertex. Thus,  $\sigma$  has as many cycles as  $M$  has black vertices, and the degree of each vertex  $b$  is the same as the degree of the corresponding cycle  $c(b)$ .
- $\alpha$  corresponds to white vertices. Just as above for each white vertex  $w \in W$  of  $M$ , we define  $c(w)$  to be the cycle consisting of the labels on the germs incident to the white vertex  $w$  taken in the positive direction around the vertex. If  $W = \{w_1, \dots, w_n\}$  is the set of white vertices, then  $\alpha = c(w_1) \cdot c(w_2) \cdot \dots \cdot c(w_n)$  is the product of the permutations corresponding to each white vertex.
- $\psi$  corresponds to face. Let  $f \in F$  be a face of  $M$ . A germ is said to be incident to a face if its label is inside the face.(note that the number of germs incident to a face is equal to its face degree i.e. half the number of germs bordering the face). Then  $c(f)$  is the cycle consisting of the labels of the germs incident to the face  $f$  taken in the positive direction from inside the face.



**Example 1.1.** For the hypermap of figure 1, We obtain the following permutations: (We also show the cycles of length 1 which correspond to vertices or faces of degree 1).

$$\begin{aligned}\sigma &= (1, 2, 3)(4, 5)(6)(7, 8, 9), \\ \alpha &= (1, 4)(2, 9, 3)(5, 6, 7)(8), \\ \phi &= (1, 5, 9)(2)(3, 8, 7, 6, 4).\end{aligned}$$

For the outer face, one may think that the corresponding cycle  $(3, 8, 7, 6, 4)$  turns in the negative direction, but in fact, we must look "from the opposite side of the sphere", or, in other words "from the inside of the outer face", and then the direction becomes positive.

The following is true for any hypermap:

$$\sigma\alpha\psi = 1.$$

(This is the standard notation used in the literature, but beware in this notation we describe functions composing from left to right, so explicitly if we apply  $\sigma$  then  $\alpha$  then  $\psi$ , we get the identity. In more standard composition notation  $\psi\alpha\sigma = 1$ .)

The proof is a simple exercise, just track any particular label through the following permutations.

**Proposition 1.2.** *To any triple of permutations  $(\sigma, \alpha, \psi)$  such that:*

- 1 *The permutation graph  $G = \langle \sigma, \alpha, \psi \rangle$  generated by  $\sigma, \alpha, \psi$  is transitive.*
- 2  *$\sigma\alpha\psi = 1$  (in permutation language) or equivalently  $\psi\alpha\sigma = 1$  (in function composition language)*

*There corresponds a hypermap.*

Transitivity ensures that the corresponding hypermap is connected.

**Definition 1.3** (Cartographic Group). *Given a hypermap  $M$  of degree  $m$ , with corresponding permutations  $\sigma, \alpha, \psi$ , we call the group  $G = \langle \sigma, \alpha, \psi \rangle \subset S_m$  the cartographic group of  $M$ .*

**Observation 1.4.** *It will very often turn out that the cartographic group of a hypermap is either  $A_m$  or  $S_m$  [1] but there are many more interesting cartographic groups.*

**Definition 1.5** (Ramification Index). *Let  $S$  and  $S'$  be Riemann Surfaces. Let  $\pi : S' \rightarrow S$  be a ramified cover. Then the ramification index of a point  $p \in S'$  is the unique  $n$  such that there exists local coordinates around  $p$  and around  $\pi(p)$  such that  $\pi(z) = z^n$ .*

**Definition 1.6** (Belyi Function). *Let  $X$  be a Riemann Surface. A ramified cover  $f : X \rightarrow \overline{\mathbb{C}} = \mathbb{P}_{\mathbb{C}}$  is a Belyi function if it is only ramified above the points  $\{0, 1, \infty\}$ . if  $f$  is a Belyi then we construct the hypermap corresponding to  $X$  as follows.*

- 1 *The black vertices of  $M$  are exactly the points  $x \in X$  such that  $f(x) = 0$ , and the multiplicity of each vertex is equal to its ramification index under  $f$ .*
- 2 *The white vertices of  $M$  are exactly the points  $x \in X$  such that  $f(x) = 1$ , and the multiplicity of each vertex is equal to its ramification index under  $f$ .*
- 3 *Inside each face, there exists a single point  $x$  with  $f(x) = \infty \in \overline{\mathbb{C}}$ , and the ramification index of this point is equal to the degree of the face.*
- 4 *The hypermap  $M$  is obtained by  $M = f^{-1}([0, 1])$ .*

**Definition 1.7** (Belyi Pair). A pair  $(X, f)$  where  $X$  is a Riemann surface, and  $f : X \rightarrow \overline{\mathbb{C}}$  is a meromorphic function, is called a Belyi pair if  $f$  is a Belyi map. (i.e. all the critical points of  $f$  are preimages of  $\{0, 1, \infty\}$ ).

**Theorem 1.8.** For any hypermap  $M$  on a surface  $X$ , there exists a Belyi pair  $(X, f)$  such that  $M$  is embedded in  $X$  as  $M = f^{-1}([0, 1])$ .

## 2. RIEMANN SURFACES

A Riemann Surface is a one dimensional Complex manifold with biholomorphic transition functions. For our purposes, however, when we speak of a Riemann Surface, we will always be referring to a particular choice of embedding of that Riemann Surface into  $\mathbb{CP}^n$ .

In  $m$  dimensional affine space  $\mathbb{C}^m$ , let  $f_1, \dots, f_n$  be polynomials in the complex variables  $x_1, \dots, x_m$ . Then the set of points  $(a_1, \dots, a_m) \in \mathbb{C}^m$  such that  $f_1(a_1, \dots, a_m) = \dots = f_n(a_1, \dots, a_m) = 0$  is called the vanishing set of these polynomials or the variety defined by these polynomials. This set of points is denoted  $V(f_1, \dots, f_n) = \{(a_1, \dots, a_m) \in \mathbb{C} : f_1(a_1, \dots, a_m) = \dots = f_n(a_1, \dots, a_m) = 0\}$ .

In  $m$  dimensional projective space  $\mathbb{P}_{\mathbb{C}}^m$ , we define projective varieties similarly. Let  $g_1, \dots, g_n$  be homogeneous polynomials. Let  $x_0, \dots, x_m$  be complex variables. Then the set of points in  $[a_0 : \dots : a_m] \in \mathbb{CP}^m$  such that  $g_1(a_0, \dots, a_m) = \dots = g_m(a_0, \dots, a_m) = 0$  is called the vanishing set of these polynomials or the variety defined by these polynomials. This set of points is denoted  $V(g_1, \dots, g_n) = \{[a_0 : \dots : a_m] \in \mathbb{CP}^n : g_1(a_0, \dots, a_m) = \dots = g_n(a_0, \dots, a_m) = 0\}$ .

For the purpose of dessins, we will be dealing with Riemann surfaces that are embedded as varieties in complex projective space. Define the jacobian matrix  $Jac(V)$  of the variety  $V(f_1, \dots, f_n)$  by  $Jac(V)_{ij} = \frac{\partial f_i}{\partial x_j}$ .

If the for each point  $X$  in our variety  $V$  consider  $Jac(V)(X)$ . Note that the kernel of  $Jac(V)(X)$  looks like the tangent space of the variety  $V$  at  $X$  in the case of a suitably nice variety  $V$ . This makes the following definitions more sensible.

**Definition 2.1.** The variety  $V$  is smooth if for every  $X \in V$ , the kernel of  $Jac(V)(X)$  has the same dimension.

**Definition 2.2.** The dimension of a smooth variety  $V$  is the dimension of the kernel of  $Jac(V)(X)$  at every point  $X \in V$

If our variety is not smooth, then we define the dimension in terms of the functions on our variety.

**Definition 2.3.** The dimension of a variety  $V(f_1, \dots, f_n) \in \mathbb{C}^m$  is the transcendence degree of  $\mathbb{C}[x_1, \dots, x_m]/(f_1, \dots, f_n)$  over  $\mathbb{C}$ .

**Definition 2.4.** a point  $X$  in a variety  $V$  is smooth if the kernel of  $Jac(V)(X)$  has the same dimension as the variety  $V$ .

When we speak of a Riemann Surface, we will mean a one dimensional smooth variety in  $\mathbb{CP}^n$ . We choose such concrete examples of Riemann Surfaces because we are interested in talking very concretely about the functions on them in order to place dessins on these Surfaces. We lose no generality by Belyi's Theorem.

**Theorem 2.5** (Belyi's Theorem).  $C$  is a nonsingular algebraic complex curve defined by algebraic coefficients if and only if there is a Belyi map  $f : C \rightarrow \mathbb{P}_{\mathbb{C}}^1$

Proven in [3]

### 3. COMPUTING BELYI MAPS

We will give a short overview of a few techniques for computing Belyi Maps, along with examples.

First note that a Belyi maps on the plane are defined up to precomposition with a biholomorphic map from the plane to itself i.e. a fractional linear transformation  $x \mapsto \frac{ax+b}{cx+d}$ . This precomposition allows us to place 3 vertices of our dessin.

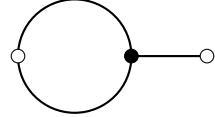
The most direct way to compute a Belyi Map is by simply solving an algebraic system of equations, but these systems can be very large, groebner bases and Hensel lifting can be used to solve large systems of equations, but first let us review some tricks.

first to be clear what we are trying to do when we compute a Belyi Map or place a dessin let us look at the following dessin.



The corresponding Belyi Map is  $f(z) = z^2$ , because this dessin is the preimage of the interval  $[0, 1]$  under  $f$  where preimages of 0 are black vertices and preimages of 1 are white vertices. Because  $z^2$  has a zero of order 2 at 0 the black vertex is degree 2. Because  $z^2 - 1$  has zeros of degree 1 at  $\pm 1$ , the white vertices are degree 1.

**Example 3.1.** let us find the Belyi Map corresponding to the following dessin in the plane.



There is one black vertex of order 3 and two faces of order 1 and 2 respectively. Thus, we have an equation of the form  $f(z) = k \frac{(x-c)^3}{(x-d)^2(x-e)}$ .

First, we can place 3 points due to the freedom given by precomposing with a fractional linear transformation. let us place the black vertex at the origin, the center of the exterior face at  $\infty$  and the center of the other face at  $-1$ .

Then we have our Belyi map  $f(x) = k \frac{x^3}{x+1}$  (note that terms of the form  $x - \infty$  vanish from the equation).

and because of the degrees of the two white vertices, we have  $f(x)-1 = k \frac{(x-a)(x-b)^2}{x+1}$ . Where  $b$  is the position of the degree 2 white vertex and  $a$  is the position of the degree 1 white vertex.

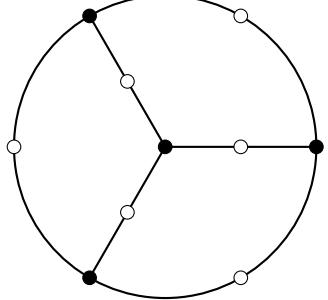
At this point we could multiply out  $k \frac{x^3}{x+1} - 1 = k \frac{(x-a)(x-b)^2}{x+1}$  and set coefficients equal to solve algebraically, but instead, we will work with derivatives.

Now the derivative of  $f(x)$  will be the same as the derivative of  $f(x) - 1$ . It will have each zero of  $f$  with its degree reduced by 1 and each zero of  $f(x) - 1$  with its degree reduced by 1 (just by quotient rule).

Taking the derivative of  $f$  gives  $f'(x) = \frac{2x^2(x+3/2)}{(x+1)^2}$ . Thus  $b = -3/2$ . Now we want to choose  $k$  such that  $f(-3/2) - 1 = 0$ . Evaluating gives  $k = \frac{4}{3^3}$ . So our Belyi Map is  $f(x) = \frac{4}{3^3} \frac{x^3}{x+1}$ .

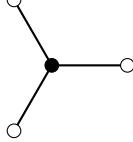
Note how we took the derivative of  $f$  to find  $b$ . Taking derivatives often allows us to simplify the system of algebraic equations we need to solve.

**Example 3.2.** Now let us compute the Belyi Map corresponding to the tetrahedron by using symmetry , which in the case of dessins corresponds to function composition. We seek to find the Belyi Map corresponding to this dessin:



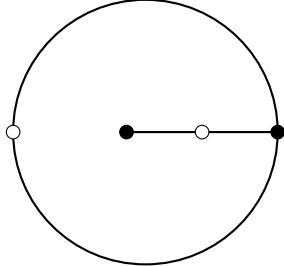
Note that if we place the central black vertex at the origin, this figure has three-fold rotational symmetry around the origin.

This means that we can compose a degree 4 dessin with this dessin



corresponding to  $f(z) = z^3$  to get our desired threefold symmetry.

We will choose the below dessin which looks like a third of the tetrahedron.



It corresponds to the Belyi map  $g(z) = -\frac{1}{64} \frac{z(z-8)^3}{(z+1)^3}$ .

When we compose the two we get  $g \circ f(z) = -\frac{1}{64} \frac{z^3(z^3-8)^3}{(z^3+1)^3}$  which corresponds to our tetrahedron.

Instead of computing a dessin of order 12 we used this symmetry to compute a dessin of order 3 and a dessin of order 4, which is much easier, because so finding a dessin directly by solving algebraic equations increases in difficulty very quickly with dimension.

Alexander Zvonkin goes through this theory in [7], and gives many examples in [5]. We now give the general theorem on when such compositions are possible.

**Definition 3.3** (imprimitive group). A permutation group  $P \subset S_n$  is imprimitive if there exists a partition of  $1, \dots, n$  into nontrivial blocks  $D_1, \dots, D_m$  such that the image of each block when acted on by any element of  $P$  is a block. (nontrivial means each  $1 < |D_i| < n$ ).

**Theorem 3.4** (Ritt's Theorem). An unramified covering  $h : X \rightarrow Z$  of degree  $\deg(h) = mn$  is a composition of two coverings  $h = g \circ f$  where  $f : X \rightarrow Y$  and

$g : Y \rightarrow Z$  with  $\deg(f) = m$  and  $\deg(g) = n$  if and only if the monodromy group of  $H$  is imprimitive and has  $n$  blocks of size  $m$ .

Proved in [7]

The reason we are interested in this result is that in the case of Belyi maps the cartographic group is the monodromy group, so a Belyi map  $h$  can be described as a composition of Belyi maps  $f$  and  $g$  if and only if the cartographic group of  $h$  is imprimitive.

#### 4. EXAMPLES ON A TORUS

Let us start placing dessins on a torus with branch cuts. We will begin with the dessin corresponding to the map  $f(z) = z^2$  in the plane.

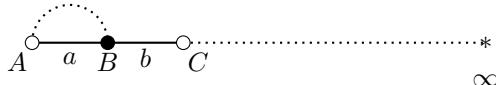


Now if we place branch cuts from  $-1$  to  $0$  and from  $1$  to  $\infty$ , the degree of each of our vertices and our face doubles, because they are all branch points, let's compute the resulting dessin.



FIGURE 1. Dessin  $f(x) = x^2$  with branch cuts from  $-1$  to  $0$  and  $1$  to  $\infty$ .

We have two sheets now to see our final dessin let us first label our vertices  $A, B, C$  and our edges  $a, b$ .



Now we look at the order in which edges leave our vertices remembering that as we circle a vertex crossing a branch cut switches which sheet we are on.

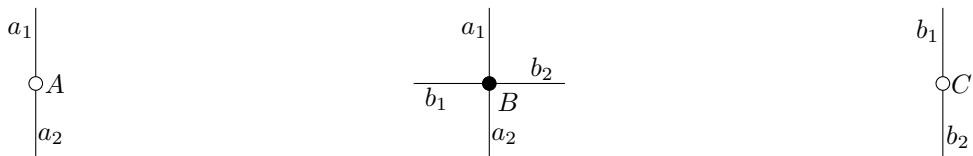


FIGURE 2. In this figure  $a_1$  is the edge  $a$  on sheet 1, and  $a_2$  is the edge  $a$  on sheet 2. We do not place a number next to the vertices in this case because they are branch points and thus are on both sheets.

Call our torus  $C$ . Our branch points were  $0, 1, -1, \infty$ , so the equation defining  $C$  is  $y^2 = x(x-1)(x+1) = x^3 - x$ . The map  $f : C \rightarrow \mathbb{P}_{\mathbb{C}}^1$  defined by  $(x, y) \mapsto x^2$  gives the dessin in Figure 3.

Now we will compose this simple dessin on a torus with other Belyi maps to place  $K_4$  and  $K_5$  on a torus. We consider two maps, we will call the Belyi maps

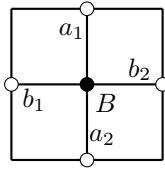


FIGURE 3. The above box is a torus with opposite sides identified. Putting this dessin together gives the simplest dessin on a torus (Weierstrass  $\wp$ -function squared). The Belyi Map corresponding to this dessin is  $(x, y) \mapsto x^2$  on the curve  $y^2 = x(x - 1)(x + 1)$ .

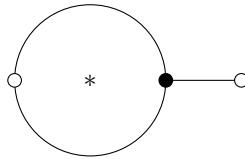


FIGURE 4. Truncation dessin corresponding to the Belyi map  $f_{trunc}$ .

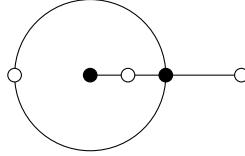


FIGURE 5. Pointed truncation dessin corresponding to the Belyi map  $f_{ptrunc}$ .

corresponding to the dessins in Figure 4 and Figure 5, the truncation map, and the pointed truncation map respectively. We will compute these Belyi maps in a moment.

Truncating the vertices of a map means replacing each black vertex with a face of the same degree such that in the cycle around that face each black vertex has degree 3.  $f_{trunc}$  is called the truncation map, because composing a Belyi map  $g$  with  $f_{trunc}$  to get  $f_{trunc} \circ g$  truncates the vertices of the map corresponding to  $g$ .  $f_{trunc}$  was used heavily in [5] to compute maps corresponding to truncated archimedean solids. I have not seen  $f_{ptrunc}$  used before, but it is similar to  $f_{trunc}$ . In the diagrams below, all white vertices are order 2, so they are suppressed.

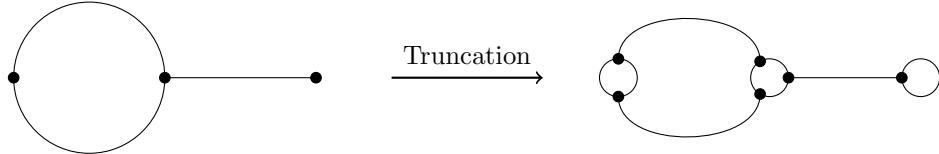
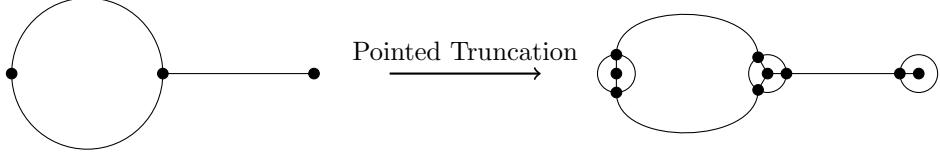


FIGURE 6. On left a dessin  $g$ . On right  $f_{trunc} \circ g$ .

Let us compute  $f_{trunc}$ . We want to place the internal face at 0, the white vertex of degree 1 at 1 and the external face at  $\infty$ . Thus we get the equations

FIGURE 7. On left a dessin  $g$ . On right  $f_{ptrunc} \circ g$ .

$$f_{trunc}(x) = k \frac{(x-b)^3}{x}$$

$$f_{trunc}(x) - 1 = k \frac{(x-a)^2(x-1)}{x}$$

taking derivatives gives

$$k \frac{2(x+b/2)(x-b)^2}{x^2} = \frac{d}{dx} f_{trunc}(x) = \frac{d}{dx} (f_{trunc}(x) - 1) = k \frac{(x-a)(2x^2-x-a)}{x^2}$$

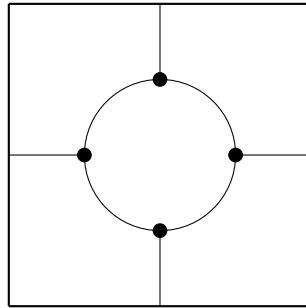
Thus we get that  $b/2 = -a$  and  $2x^2 - x - a = 2(x-b)^2$ . The polynomial  $2x^2 - x - a$  is only a square if the discriminant  $(-1)^2 - 4 * 2 * (-a) = 0$ . So  $a = -\frac{1}{8}$ , and  $b = 1/4$ . So we get that  $f_{trunc}(x) = k \frac{(x-1/4)^3}{x}$ . We want  $f(b) = 1$  so we get  $1 = f(-1/8) = k(27/64)$  so  $k = 64/27$ . Thus we have

$$f_{trunc}(x) = (64/27) \frac{(x-1/4)^3}{x}$$

The same techniques can be used to find the Belyi Map corresponding to  $f_{ptrunc}$ , but because it is degree 5 the equations take a bit longer to solve so we will omit the computation and simply give the Belyi Map.

$$\text{Let } c = \frac{-480 \pm \sqrt{480^2 - 4(200)(148)}}{-296} \text{ Let } a = 2c(\frac{10c}{16c-10}) \text{ Let } k = \frac{(\frac{10c}{16c-10} - 1)^3}{(\frac{10c}{16c-10})^4((2c-1)(\frac{10c}{16c-10}))}$$

$$f_{ptrunc}(x) = k \frac{x^4(x-a)}{(x-1)^3}$$

FIGURE 8. The Belyi Map  $f_{trunc} \circ g = f_{trunc}(x^2)$  places  $K_4$  on the torus defined by  $y^2 = x^3 - x$ .

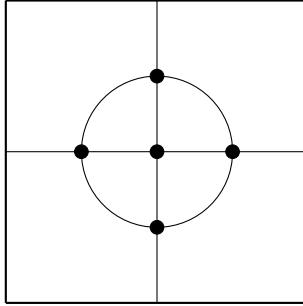


FIGURE 9. The Belyi Map  $f_{ptrunc} \circ g = f_{ptrunc}(x^2)$  places  $K_5$  on the torus defined by  $y^2 = x^3 - x$ .

### 5. BLOWING UP A POINT

Several of the curves we define in the following sections will be singular, in particular, they will self intersect. We will resolve this by Blowing up the points of self-intersection until we reach a nonsingular curve.

We will first describe what it means to blow up the origin in  $\mathbb{A}^n$ . Consider  $\mathbb{A}^n$  with affine coordinates  $(x_1, \dots, x_n)$ , and introduce projective coordinates  $[t_1, \dots, t_n]$ . Consider the sub variety  $B_0$  of  $\mathbb{A}^n \times \mathbb{P}^{n-1}$  defined by the equations  $x_i t_j - x_j t_i$  for all  $i, j$ . The projection map  $\pi : B_0 \rightarrow \mathbb{A}^n$  defined by  $(x_1, \dots, x_n, t_1, \dots, t_n) \mapsto (x_1, \dots, x_n)$  is injective except at the preimage of the origin. with  $(x_1, \dots, x_n, x_1, \dots, x_n) \mapsto (x_1, \dots, x_n)$ . At the preimage of the origin  $x_i = 0$  for all  $i$  so all the polynomials defining  $V$  are 0. Thus the preimage of the origin is an entire projective space  $\{0\} \times \mathbb{P}^{n-1}$ .

To blow up any other point  $p$  we simply change coordinates such that  $p$  lies at the origin. To blow the point  $[1 : 0 : \dots : 0]$  in projective space  $\mathbb{P}^n$  we consider  $\mathbb{A}^n \subset \mathbb{P}^n$  by identifying  $(x_1, \dots, x_n)$  with  $(1 : x_1 : \dots : x_n)$ . Blow up the origin of  $\mathbb{A}^n$  to get  $B_0 \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ . Now glue together  $B_0$  and  $\mathbb{P}^n \setminus [1 : 0 : \dots : 0]$  by identifying  $((x_1, \dots, x_n), [x_1, \dots, x_n]) \in B_0$  with  $[1 : x_1 : \dots : x_n]$  whenever  $(x_1, \dots, x_n) \neq (0, \dots, 0)$ . The resulting variety  $B$  is the blowup of  $\mathbb{P}^n$  at  $[1 : 0 : \dots : 0]$ .

If  $Y$  is a sub variety of  $\mathbb{A}^n$  or  $\mathbb{P}^n$  then the blowup of  $Y$  at the origin or  $[1 : 0 : \dots : 0]$  respectively is the closure in  $B_0$  of  $\pi^{-1}(Y \setminus \{0\})$ , where  $B_0$  is the blowup of  $\mathbb{A}^n$  at the origin, or the blowup of  $\mathbb{P}^n$  at  $[1 : 0 : \dots : 0]$  respectively. Blowups will help us move from singular varieties to nonsingular varieties due to the following theorem.

**Theorem 5.1.** *Any variety  $X$  is birationally equivalent to a smooth variety  $Y$ .  $X$  can be reached from  $Y$  by blowing up sub varieties .*

Proven in [2]

In the case of a complex curve the only sub varieties are finite collections of points. So given a singular curve, we can blow-up points until it is smooth.

### 6. COMPUTING A DESSIN THAT PLACES $K_{n,n}$ ON A SURFACE.

Let  $C_{n,n} \subset \mathbb{P}_{\mathbb{C}}^2$  considered with coordinates  $[x : y : z]$  be the curve defined by  $y^n = x^n(x^n - 1)$  and let  $f_{n,n} : C_{n,n} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  be the map defined by  $(x, y) \mapsto x^n$ . Note that  $C_{n,n}$  is singular due to self-intersections. Blow up the points of self-intersection until this curve is no longer singular. Denote the nonsingular curve

obtained in this way by  $\overline{C_{n,n}}$ . Let  $\pi : \overline{C_{n,n}} \rightarrow C_{n,n}$  be the map which projects each point of  $\overline{C_{n,n}}$  onto its  $[x : y : z]$  coordinates. Let  $\overline{f_{n,n}} : \overline{C_{n,n}} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  be the map defined by  $\overline{f_{n,n}} = f_{n,n} \circ \pi$ .

**Theorem 6.1.**  $\overline{f_{n,n}} : \overline{C_{n,n}} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is a Belyi Map which places the dessin  $K_{n,n}$  on the curve  $\overline{C_{n,n}}$  of genus  $\frac{n^2-3n+2}{2}$ .

*Proof.* The function  $f_{n,n} : C_{n,n} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is the composition  $(\pi_x)^n$ . Since  $\pi_x$  is ramified only above  $\{0, 1, \infty\}$  and the map  $x \mapsto x^n$  is unramified outside  $\{0, 1, \infty\}$ , the function  $f_{n,n}$  is unramified except above  $\{0, 1, \infty\}$ . Finally because  $\pi : \overline{C_{n,n}} \rightarrow C_{n,n}$  is unramified except over points in  $C_{n,n}$  with x-values  $\{0, 1, \infty\}$ , we get that  $\overline{f_{n,n}} = f_{n,n} \circ \pi$  is unramified except over the points  $\{0, 1, \infty\}$ . So  $\overline{f_{n,n}}$  is a Belyi Map.

Now let us prove that the dessin corresponding to this Belyi Map is an embedding of  $K_{n,n}$ . First note that  $\overline{f_{n,n}}$  is a cover of degree  $n^2$ . So if it were generic, we would expect the preimage of the interval  $[0, 1]$  to look like figure 15.

If we can determine how the preimages of 0 and 1 ramify under the map  $\overline{f_{n,n}}$ , then we have determined the dessin corresponding to  $\overline{f_{n,n}}$ . This ramification data can be found by pulling back small loops around 0 and 1 by  $\overline{f_{n,n}}$  to get paths in  $\overline{C_{n,n}}$ . Note however that because  $C_{n,n} \setminus f_{n,n}^{-1}(\{0, 1, \infty\})$  is isomorphic to  $\overline{C_{n,n}} \setminus \overline{f_{n,n}}^{-1}(\{0, 1, \infty\})$ , under the restriction of  $\pi : \overline{C_{n,n}} \rightarrow C_{n,n}$ , and that  $\overline{f_{n,n}} = f_{n,n} \circ \pi$ , we can obtain the same data by pulling our loops around the points 0 and 1 in  $\mathbb{P}_{\mathbb{C}}^1$  back under  $f_{n,n}$  to obtain paths in  $C_{n,n}$ .

Let  $\epsilon > 0$  be a real number. To track the ramification above 0, consider a very small circle around the origin in  $\mathbb{P}_{\mathbb{C}}^1$  defined by the equation  $\alpha(t) = \epsilon e^{2\pi i t}$ . Choose one of the  $n^2$  preimages of  $\epsilon$  under  $f_{n,n}$  and call it  $(x_0, y_0)$ . The set of preimages of  $\epsilon$  is  $\{(\zeta_n^i x_0, \zeta_n^j y_0) | 0 \leq i, j < n\}$  where  $\zeta_n = e^{\frac{2\pi}{n} i t}$ . Let  $\beta_{i,j}(t)$  be the pull back of  $\alpha(t)$  such that  $\alpha(0) = (\zeta_n^i x_0, \zeta_n^j y_0)$ , then  $\beta_{i,j}(1) = (\zeta_n^{i+1} x_0, \zeta_n^{j+1} y_0)$ . This can be seen by setting  $x^n = e^{2\pi i t}$  and setting  $y^n = x^n(x^n - 1)$  then tracking  $x$  and  $y$  as  $t$  ranges from 0 to 1.

Tracking the ramification above 1 is similar. Consider a very small circle around 1 in  $\mathbb{P}_{\mathbb{C}}^1$  defined by the equation  $a(t) = 1 + \epsilon e^{2\pi i t}$ . Choose one of the  $n^2$  preimages of  $1 - \epsilon$  under  $f_{n,n}$  and call it  $(x_1, y_1)$ . The set of preimages of  $1 - \epsilon$  is  $\{\zeta_n^i x_1, \zeta_n^j y_1 | 0 \leq i, j < n\}$ . Let  $b_{i,j}(t)$  be the pullback of  $a(t)$  under  $f_{n,n}$  such that  $b_{i,j}(0) = (\zeta_n^i x_1, \zeta_n^j y_1)$ , then  $b_{i,j}(1) = (\zeta_n^i x, \zeta_n^{j+1} y)$ . This can be checked by setting  $x^n = 1 - \epsilon e^{2\pi i t}$ , setting  $y^n = x^n(x^n - 1)$  and tracking  $x$  and  $y$  as  $t$  goes from 0 to 1.

Finally observe that in terms of ramification data, a counterclockwise path around  $\infty$  is equivalent to a clockwise path around both 0 and 1. Let  $c : [0, 1] \rightarrow \mathbb{P}_{\mathbb{C}}^1$  be a clockwise loop around 0 and 1. Choose one preimage of  $c(0)$  under  $\overline{f_{n,n}}$ , and call it  $(x_{\infty}, y_{\infty})$ . The set of all preimages of  $c(0)$  under  $\overline{f_{n,n}}$  is  $\{(\zeta_n^i x, \zeta_n^j y) | 0 \leq i, j < n\}$ . Let  $c_{i,j}(t)$  be the pullback of  $c(t)$  such that  $c_{i,j}(0) = (\zeta_n^i x, \zeta_n^j y)$ , then because we understand the ramification at 0 and 1 we see that  $c_{i,j}(1) = (\zeta_n^{i-1} x, \zeta_n^{j-2})$ . All of this can be captured in the following statements:

Ramification at 0: If the equation  $y^n = x^n(x^n - 1)$  is satisfied, then as  $x^n$  goes around 0 once:

- $x$  makes  $\frac{1}{n}$ th a rotation around 0 ( $x \mapsto \zeta_n x$ ).

- $y^n$  makes a full rotation around 0.
- Hence  $y$  makes  $\frac{1}{n}$ th a rotation around 0 ( $y \mapsto \zeta_n y$ ).

Ramification at 1: If the equation  $y^n = x^n(x^n - 1)$  is satisfied, then as  $x^n$  goes around 1 once:

- $x$  makes a full rotation around  $-1$  or  $1$  ( $x \mapsto x$ ).
- $y^n$  makes a full rotation around 0.
- Hence  $y$  makes half a rotation around 0 ( $y \mapsto \zeta_n y$ ).

Ramification at  $\infty$ : Hence If the equation  $y^n = x^n(x^n - 1)$  is satisfied, then as  $x^n$  goes around  $\infty$  or equivalently goes around both 0 and 1:

- The pullback of a path around  $\infty$  that starts at  $x$  ends at  $\zeta_n^{-1}x$ .
- The pullback of a path around  $\infty$  that starts at  $y$  ends at  $\zeta_n^{-2}y$ .

Due to the ramification at  $\infty$ , we see that when this graph is embedded on  $\overline{C_{n,n}}$  it has  $n$  faces. So we can compute the genus by setting  $2-2g = V-E+F = 2n-n^2+n$ . So  $g = \frac{n^2-3n+2}{2} = \frac{(n-1)(n-2)}{2}$ .

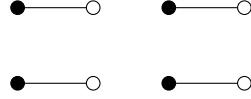


FIGURE 10. For  $n = 2$ . The preimage of  $[0, 1]$  under a generic degree 4 function.

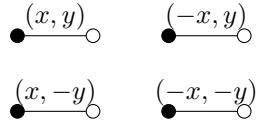


FIGURE 11. For  $n=2$ . Above we chose  $(x_0, y_0)$  and  $(x_1, y_1)$  each to lie on the same preimage of the interval  $[0, 1]$  and name this interval  $(x, y)$ . We then label other segments based on which of  $\{\pm x_0, \pm y_0\}$  lie on the segment.



FIGURE 12. For  $n=2$ . When we pull back a loop around 0 or 1 starting at a point  $(x, y)$  the loop ends at the point  $(x, y) \times (-1, -1)$  or  $(x, y) \times (1, -1)$  respectively.

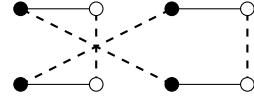


FIGURE 13. For  $n=2$ . The dotted lines connect points which are actually glued together into a single ramification point in the case where we are looking at the preimage of  $[0, 1]$  under  $\overline{f_{2,2}}$ .

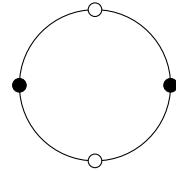


FIGURE 14. For  $n=2$ . If we glue together the points connected by dotted lines in 13 we get a hypermap corresponding to  $K_{2,2}$ .

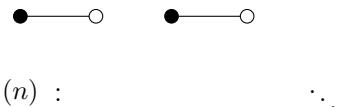
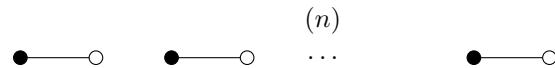


FIGURE 15. Preimage of  $[0, 1]$  under a generic  $n^2$  sheeted cover.

$$(\zeta_n, \zeta_n) \quad (1, \zeta_n)$$

FIGURE 16. When we pull back a loop around 0 or 1 respectively starting at a point  $(x, y)$  the loop ends at the point  $(x, y) \times (\zeta_n, \zeta_n)$  or  $(x, y) \times (1, \zeta_n)$  respectively.

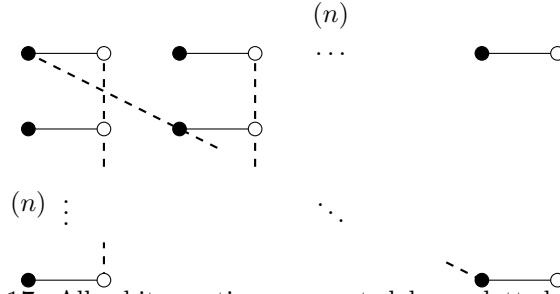


FIGURE 17. All white vertices connected by a dotted line glue together into a single white vertex at a ramification point. All black vertices connected by a dotted line glue together to form a single black vertex at a ramification point. Because the dotted lines connecting white vertices and the dotted lines connecting black vertices are not parallel they each pair of dotted lines connecting different color vertices have one sheet in common. So when we glue together every black vertex is connected to every white vertex exactly once, so the resulting graph is  $K_{n,n}$ .

□

## 7. CAYLEY GRAPHS ON THE GROUP $\mathbb{Z}_2^n$

**Definition 7.1.** A Cayley Graph  $(G, S)$  of a group  $G$  with a generating set  $S$  is the graph whose vertex set  $V$  is the set of elements of  $G$  and whose edge set is  $E = \{(x, y) : (x, y) \in V \times V, x - y \in S\}$ .

Let  $G$  be the group  $\mathbb{Z}_2^n$  for some  $n$ . Let  $S = \{s_1, \dots, s_k\}$  be a subset of  $G$  such that each element  $s_r$  is a tuple of length  $n$ . Let  $s_r(i)$  be the  $i$ th element of the tuple. We will construct a Belyi Map which places the Cayley Graph  $(G, S)$  as a dessin on a surface. Let  $\zeta_k = e^{\frac{2\pi}{k}i}$  be a  $k$ th root of unity. Let  $v : \{0, 1\} \rightarrow \{1, 2\}$  be the function defined by  $v(0) = 2$  and  $v(1) = 1$ . Then let  $C_{(G,S)}$  be the variety in  $\mathbb{P}_{\mathbb{C}}^{n+1}$  in the variables  $[x : y_1 : \dots : y_n : z]$  defined by the polynomials  $y_i^2 = \prod_{r=1}^k (x - \zeta_k^{s_r(i)})^{v(s_r(i))}$ . Let  $f_{(G,S)} : C_{(G,S)} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  be the map defined by  $[x : y_1 : \dots : y_n : z] \mapsto [x^k : z]$  (note that if both  $x$  and  $z$  are zero, then every  $y_i$  is equal to zero, so this is a well defined map). Now  $C_{(G,S)}$  is a singular curve due to self-intersections at  $(\zeta_k^r, 0)$  for some  $k$ th roots of unity  $\zeta_k^r$ . Blow up these singular points until the curve is no longer singular, and call the resulting curve  $\overline{C_{(G,S)}}$ . Let  $\pi : \overline{C_{(G,S)}} \rightarrow C_{(G,S)}$  be the projection onto the  $[x : y_1 : \dots : y_n : z]$  coordinates. Then define  $\overline{f_{(G,S)}} : \overline{C_{(G,S)}} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  by  $\overline{f_{(G,S)}} = f_{(G,S)} \circ \pi$ .

**Theorem 7.2.**  $\overline{f_{(G,S)}} : \overline{C_{(G,S)}} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is a Belyi Map, which places the dessin corresponding to the Cayley Graph  $(G, S)$  on the surface  $\overline{C_{(G,S)}}$ . The surface  $\overline{C_{(G,S)}}$  has genus  $g = \frac{k \times 2^{n-1} - 2^n - lcm(2, n) + 2}{2}$ .

*Proof.* As in the case of computing the dessin for  $K_{n,n}$  we argue that  $\overline{f_{(G,S)}}$  is a Belyi Map, because  $f_{(G,S)}$  is ramified or singular only above  $\{0, 1, \infty\}$  and  $\pi$  is ramified only above singular points of  $f_{(G,S)}$ , so  $\overline{f_{(G,S)}} = f_{(G,S)} \circ \pi$  is ramified only above  $\{0, 1, \infty\}$ .

As in the  $K_{n,n}$  case we can find the dessin resulting from  $\overline{f_{(G,S)}}$  by understanding the ramification points in  $\overline{C_{(G,S)}}$  above the points 0 and 1. As in the case of  $K_{n,n}$  because  $\pi : \overline{C_{(G,S)}} \setminus \overline{f_{(G,S)}}^{-1}(\{0, 1, \infty\}) \rightarrow C_{(G,S)} \setminus f_{(G,S)}^{-1}(\{0, 1, \infty\})$  is an isomorphism, the ramification over 0 and 1 in  $\overline{C_{(G,S)}}$  is the same as the ramification over 0 and 1 in  $C_{(G,S)}$ , so this is the ramification we will examine.

Because each defining polynomial is of the form  $y_i^2 = \prod_{r=1}^k (x - \zeta_k^r)^{v(s_r(i))}$ , the map  $\pi : (x, y) \mapsto x$  is unramified above 0, so we need only check the ramification of  $\pi_x$  at the  $k$ th roots of unity  $\zeta_k^r$ .

Let  $\epsilon > 0$  be a small real number. Choose one of the  $2^n$  preimages of  $\zeta_k - \epsilon$  under the map  $\pi_x : C_{(G,S)} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  and call it  $(x_k, (y_1)_k, \dots, (y_n)_k)$ . The set of preimages of  $\zeta_k - \epsilon$  is  $\{x_k, \pm(y_1)_k, \dots, \pm(y_n)_k\}$ . Let  $\gamma_r(t)$  be a small loop around  $\zeta_k^r$  defined by  $\gamma_r(t) = \zeta_k^r - \epsilon e^{2\pi i t}$ . For any  $B = (b_1, b_2, \dots, b_n) \in \mathbb{Z}_2^n$ , define  $\beta_{r,B}$  to be the pullback of  $\gamma_r(t)$  such that  $\beta_{r,B}(0) = (x_k, (-1)^{b_1}(y_1)_k, \dots, (-1)^{b_n}(y_n)_k)$ . Then  $\beta_{r,B}(1) = (x_k, (-1)^{b_i+s_r(i)}(y_i)_k)$ . Choose a preimage of 0 under  $f_{(G,S)}$  and call it  $(x_0, (y_i)_0)$ . Then identify each element of  $s \in \mathbb{Z}_2^n$  with the preimage  $(x_0, (-1)^{s(i)}(y_i)_0)$ . Then in the dessin resulting from the Belyi Map  $\overline{f_{(G,S)}}$ , the black vertices are identified with elements of  $\mathbb{Z}_2^n$  and two black vertices are connected exactly if they differ by some  $s_i \in S$ .  $\square$

$f : C \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is a Belyi Map, which places the dessin  $G$  on the surface  $C$ .

Ramification at 1: On the curve  $C$  defined by the above equations, when  $x^k$  goes around 1 once:

- $x$  goes around a  $k$ th root of unity once.
- When  $x$  goes around  $\zeta_k^r$  once:
  - \*  $y_i^2$  goes  $v(s_r(i))$  rotations around 0.
  - \* Hence  $y_i$  goes  $v(s_r(i))/2$  rotations around 0.

Ramification at 0: On the curve  $C$  defined by the above equations, when  $x^k$  goes around 0 once:

- $x$  goes  $1/k$ th of a rotation around 0.

Ramification at  $\infty$ : Hence on the curve  $C$  defined by the above equations, when  $x^k$  goes around  $\infty$ , or equivalently goes around both 0 and 1 in the negative direction:

- The pullback of a loop that goes around  $\infty$  and starts at  $x$  ends at  $\zeta_n^{-1}x$ .
- The pullback of a loop that goes around  $\infty$  and starts at  $y_i$  ends at  $(-1)^{s_r(i)/2}y_i$ .

The number of faces is  $n$  if  $n$  is even and  $2n$  if  $n$  is odd. Thus we can compute the genus  $2 - 2g = V - E + F = 2^n - k \times 2^{n-1} + \text{lcm}(2, n)$ . So  $g = \frac{k \times 2^{n-1} - 2^n - \text{lcm}(2, n) + 2}{2}$

**Corollary 7.3.** *By setting  $S = \mathbb{Z}_2^n \setminus \{0\}$ , we get a dessin corresponding to the complete graph  $K_{2^n}$ .*

We will show how this construction works for  $K_4$ . Let  $G = \mathbb{Z}_2^n$ , and let  $S = \{(1, 0), (1, 1), (0, 1)\}$ . Consider the space  $\mathbb{P}_{\mathbb{C}}^3$  with coordinates  $[x : y_1 : y_2 : z]$ , and let  $C$  be the variety defined by the equations  $y_1^2 = (x - 1)(x - \omega)(x - \omega^2)^2$ , and  $y_2^2 = (x - 1)^2(x - \omega)(x - \omega^2)$ . Let  $f : C \rightarrow \mathbb{P}_{\mathbb{C}}^1$  be the Belyi Map defined by  $(x, y_1, y_2) \mapsto (x^3)$ . Note that  $C$  is singular, and our eventual dessin will be defined on the nonsingular curve  $\bar{C}$  the blowup of  $C$  at all its points of self-intersection. However, we can see what our eventual dessin will be by observing the ramification of  $f$  over  $1, \omega, \omega^2$ .



FIGURE 18. The preimage of the interval  $[0, 1]$  under the function  $(\pi_x)^3$  where  $\pi_x$  is a generic 4-fold cover of  $\mathbb{P}_{\mathbb{C}}^1$ .

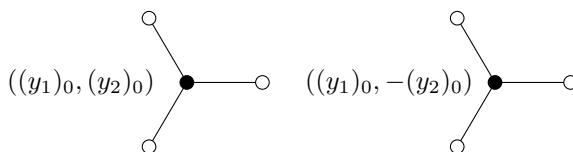


FIGURE 19. The preimage of the interval  $[0, 1]$  under the function  $(\pi_x)^3$  where  $\pi_x$  is a generic 4-fold cover of  $\mathbb{P}_{\mathbb{C}}^1$ .

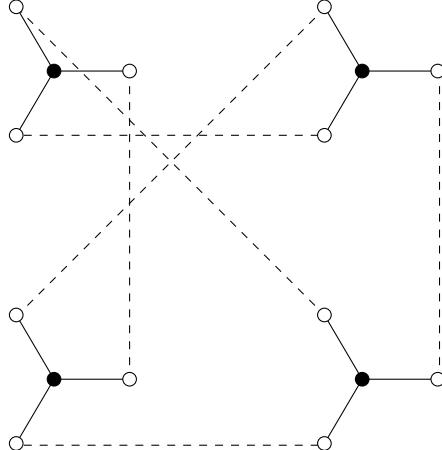


FIGURE 20. By pulling back loops around  $1, \omega, \omega^2$  onto the curve  $C$  we can see how the different sheets will glue together if  $y_1^2 = (x - 1)(x - \omega)(x - \omega^2)^2$ , and  $y_2^2 = (x - 1)^2(x - \omega)(x - \omega^2)$ . (the points connected by dotted lines will glue together).

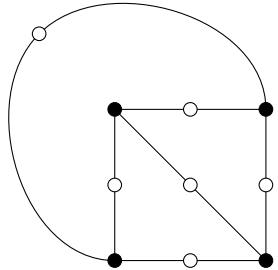


FIGURE 21. So the dessin corresponding to  $\bar{f} : \overline{C} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is this embedding of  $K_{4,4}$  in the plane.

## 8. PETERSON GRAPH

Let  $C_P \subset \mathbb{P}_{\mathbb{C}}^2$  be the curve defined by the equations  $y^5 = (x - i)(x + i)^3$ . Let  $f_P : C_P \rightarrow \mathbb{P}_{\mathbb{C}}^1$  be the map defined by  $(x, y) \mapsto x^2 + 1$ .

**Theorem 8.1.** *The map  $f_{trunc} \circ f_P : C_P \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is a Belyi Map which places the Peterson Graph as a dessin on the genus 2 surface  $C_P$ .*

*Proof.* We first compute the ramification of  $\pi_x$  at  $i$ . Let  $\epsilon > 0$  be a small real number. Choose one preimage of  $i - \epsilon$  under  $\pi_x$  on the curve  $C_P$  and call it  $(x_1, y_1)$ . The set of all preimages of  $i - \epsilon$  under  $\pi_x$  is  $\{x, \zeta_5^i y\}$ . Let  $\gamma_+(t)$  be a loop around  $i$  defined by  $\gamma_+(t) = i - \epsilon e^{2\pi i t}$ . Let  $\beta_{+,i}(t)$  be the pullback of  $\gamma_+(t)$  such that  $\beta_{+,i}(0) = (x, \zeta_5^i y)$ . Then  $\beta_{+,i}(1) = (x, \zeta_5^{i+1} y)$ . This can be seen by setting  $x = i - \epsilon e^{2\pi i t}$  and setting  $y^5 = (x - i)(x + i)^3$ .

Now we compute the ramification of  $\pi_x$  at  $-i$ . Choose one preimage of  $-i + \epsilon$  under  $\pi_x$  on the curve  $C_P$  and call it  $(x_2, y_2)$ . Then the set of all preimages of

$-i + \epsilon$  under  $\pi_x$  is  $\{(x_2, \zeta_5^i y_2)\}$ . Let  $\gamma_-(t)$  be the loop around  $-i$  defined by  $\gamma_-(t) = -i + \epsilon e^{2\pi i t}$ . Let  $\beta_{-,i}(t)$  be the pullback of  $\gamma_-(t)$ , such that  $\beta_{-,i}(0) = (x_2, \zeta_5^i y_2)$ , then  $\beta_{-,i}(1) = (x_2, \zeta_5^{i+3} y_2)$ . This can be seen by setting  $x = -i + \epsilon e^{2\pi i t}$  and setting  $y^5 = (x - i)(x + i)^3$ , and tracking this curve as  $t$  goes from 0 to 1. In the figures below we complete this construction. The information above can be summarized as follows:

Ramification at  $i$ : On the curve  $C$  defined by the equation  $y^5 = (x - i)(x + i)^3$ , when  $x$  goes around  $i$  once:

- $y^5$  goes 1 rotation around 0.
- Hence  $y$  goes  $1/5$ th of a rotation around 0.

Ramification at  $-i$ : On the curve  $C$  defined by the equation  $y^5 = (x - i)(x + i)^3$ , when  $x$  goes around  $-i$  once:

- $y^5$  goes 3 rotations around 0.
- Hence  $y$  goes  $3/5$  of a rotation around 0.

Ramification at  $\infty$ : Hence In the curve  $C$  defined by the equation  $y^5 = (x - i)(x + i)^3$ , when  $x$  goes around  $\infty$  once, or equivalently goes around  $i$  and  $-i$  in the negative direction:

- the pullback of a path around  $\infty$  that starts at  $y$  ends at  $\zeta_5 y$ .

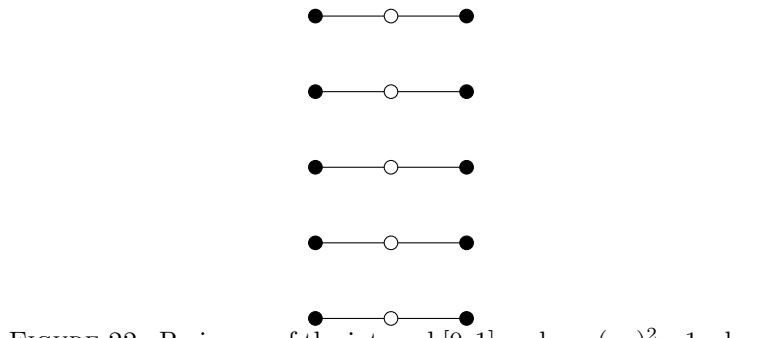


FIGURE 22. Preimage of the interval  $[0, 1]$  under a  $(\pi_x)^2 - 1$  where  $\pi_x$  is a generic 5-fold cover.

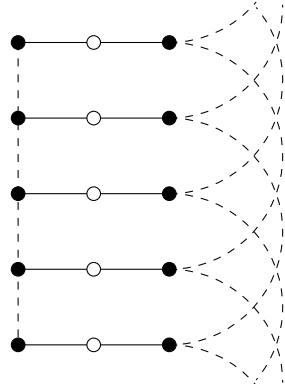


FIGURE 23. If  $\pi_x$  is projection onto the  $x$ -axis of the curve  $\bar{C}$ , then following the dotted lines downward gives the order in which points glue together to ramify.

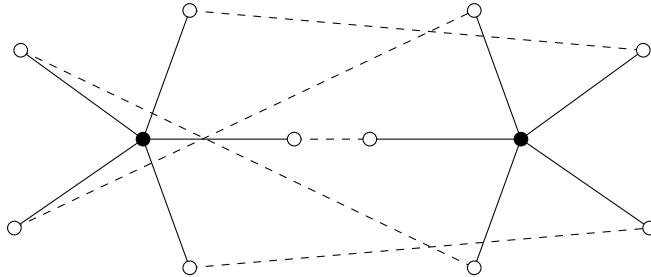


FIGURE 24. In this figure dotted lines do not represent ramification, but gluing along the dotted lines gives the dessin for  $f_P : C_P \rightarrow \mathbb{P}^1_{\mathbb{C}}$ . The same dessin as in the previous figure.

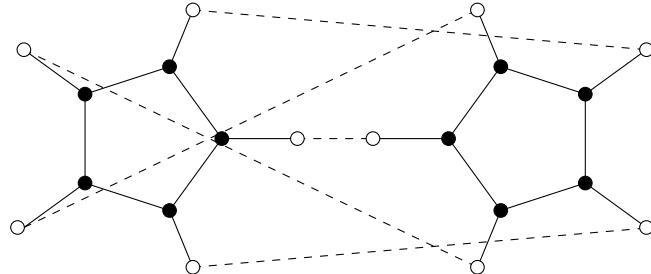


FIGURE 25. Composing with  $f_{trunc}$  gives the following dessin. It is easy to check that gluing the white vertices connected by dotted lines together gives the Petersen graph.

To see that the resulting surface is genus 2, observe that for the curve  $C$  defined by  $y^5 = (x - i)(x + i)^3$ , the path starting at a point  $(x, y)$  and circling around the point at  $\infty$  ends at  $(x, \zeta_5 y)$ . Thus the map  $f : C \rightarrow \mathbb{P}^1_{\mathbb{C}}$  defined by  $(x, y) \mapsto$

$x^2 + 1$  places a graph with 2 vertices 5 edges and 1 face on the curve  $C$ . Taking  $V - E + F = 2 - 5 + 1 = -2 = 2 - 2g$ . So  $g = 2$  for the curve  $C$ .

□

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