Linearizing Symmetric Matrix Polynomials via Fiedler pencils with Repetition

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March, 2012

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Abstract

Strong linearizations of a matrix polynomial $P(\lambda)$ that preserve some structure of $P(\lambda)$ are relevant in many applications. In this paper we characterize all the pencils in the family of the Fiedler pencils with repetition, introduced by Antoniou and Vologiannidis, associated with a square matrix polynomial $P(\lambda)$ that are symmetric when $P(\lambda)$ is. When some nonsingularity conditions are satisfied, these pencils are strong linearizations of $P(\lambda)$. In particular, when $P(\lambda)$ is a symmetric polynomial of degree k such that the coefficients of the terms of degree k and 0 are nonsingular, our family strictly contains a basis for $\mathbb{DL}(P)$, a k-dimensional vector space of symmetric pencils introduced by Mackey, Mackey, Mehl, and Mehrmann.

Acknowledgments

I would like to thank Professor Maria Isabel Bueno Cachadina for her assistance in the creation of this paper and for introducing me to this beautiful topic. This paper would not exist if it was not for her. I would also like to thank Susana Furtado for her contributions toward this paper.

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Chapter 1

Introduction

1.1 Motivation and Background Information

Definition 1 *A matrix polynomial $P(\lambda)$ of degree $k \geq 1$ is a matrix whose elements are polynomials, namely,*

$$P(\lambda) = A_k\lambda^k + A_{k-1}\lambda^{k-1} + \cdots + A_0, \tag{1.1}$$

where the coefficients A_i are $n \times n$ matrices with entries in a field $\mathbb{F}$. A matrix polynomial is said to be symmetric (Hermitian) if each A_i is symmetric (Hermitian).

Definition 2 *The* reversal *of the matrix polynomial $P(\lambda)$ in (1.1) is the matrix polynomial obtained by reversing the order of the coefficient matrices, that is,*

$$rev(P(\lambda)) := \sum_{i=0}^{k} \lambda^i A_{k-i}.$$

Matrix polynomials appear as the focus of various papers and texts as early as 1958 such as *The Theory of Matrix Polynomials and its Application to the Mechanics of Isotropic Continua* by A. J. M. Spencer, and R. S. Rivlin (21), and there are mathematical results relating to matrix polynomials from as early as 1878 (17). However, the study of matrix polynomials did not gain significant traction until the 1980s with the publication of texts such as *Matrix Polynomials* by I. Gohberg, P. Lancaster, and L. Rodman, (9), *The Theory of Matrices with Applications* by P. Lancaster and M. Tismenetsky (17), and various other texts and papers. Matrix polynomials provide useful techniques for solving problems such as polynomial eigenvalue problems in perturbed systems (3) and the study of the mechanics of isotropic continua (21). Researchers have developed more reliable and faster techniques for solving linear systems in these situations, so by reducing to a linear case while preserving relevant information can make many problems significantly easier to solve.

To simplify the study of matrix polynomials researchers developed equivalence classes and canonical forms for matrix polynomials. We will call two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ of the same size over the same field *equivalent* if there exist two unimodular matrix polynomials (i.e. matrix polynomials with constant nonzero determinant), $U(\lambda)$ and $V(\lambda)$, such that

$$U(\lambda)Q(\lambda)V(\lambda) = P(\lambda)$$

It is shown in (17) that any matrix polynomial $P(\lambda)$ is equivalent to a unique diagonal matrix polynomial

$$A(\lambda) = \mathrm{diag}[i_1(\lambda), i_2(\lambda), \ldots, i_n(\lambda)]$$

in which $i_j(\lambda)$ is zero or a monic polynomial, $j = 1, 2, \ldots, n$, and $i_j(\lambda)$ is divisible by $i_{j-1}(\lambda)$, $j = 2, 3, \ldots, n$. We call $A(\lambda)$ the *Smith canonical form* of $P(\lambda)$. H.J.S. Smith obtained this form for matrices with integer entries in 1861 and Frobenius obtained the result for matrix polynomials in 1878. The $i_j(\lambda)$ are called the *invariant factors* of $P(\lambda)$, where the invariance refers to equivalence transformations, i.e. that two matrix polynomials are equivalent iff they have the same invariant polynomials.

Because a polynomial ring over a field is a UFD we can uniquely (up to associates) factor each invariant polynomial into irreducible elements. The largest power of an irreducible element over $\mathbb{F}[X]$ that divides an invariant polynomial of $P(\lambda)$ is called a *(finite) elementary divisor* of $P(\lambda)$. An infinite elementary divisor of $P(\lambda)$ is a finite elementary divisor of $\mathrm{rev}(P(\lambda))$. It is shown in (17) that two matrix polynomials of the same size over the same field are equivalent iff they share finite elementary divisors.

Using the previous results, one can determine the Smith canonical form, eigenvalues, minimal polynomial, characteristic polynomial, invariant polynomials and more from elementary divisors. Because so much of the information about $P(\lambda)$ is stored in its elementary divisors, if one can find another matrix polynomial $Q(\lambda)$ that has the same elementary divisors but is simpler to work with, it is often beneficial to work with $Q(\lambda)$ instead of $P(\lambda)$.

Definition 3 *A matrix pencil $L(\lambda) = \lambda L_1 - L_0$, with $L_1, L_0 \in M_{kn}(\mathbb{F})$, is a* linearization *of $P(\lambda)$ (see (9)) if $L(\lambda)$ is equivalent to the matrix*

$$\begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

A linearization $L(\lambda)$ of a matrix polynomial $P(\lambda)$ is a strong linearization *when rev($L(\lambda)$) is a linearization of rev($P(\lambda)$).*

A linearization $L(\lambda)$ of a matrix polynomial $P(\lambda)$ has the same finite elementary divisors as $P(\lambda)$, and if it is a strong linearization it has the same infinite elementary divisors as well. Because $P(\lambda)$ and $L(\lambda)$ share elementary divisors we can deduce the invariant polynomials, Smith canonical form, eigenvalues, minimal polynomial, and characteristic polynomial, of $P(\lambda)$ by studying $L(\lambda)$. This is enough information to solve many of the problems in which matrix polynomials arrive. In many applications, such as the polynomial eigenvalue problem $P(\lambda)x = 0$, there are standard techniques to find exact or approximate solutions to linear systems that do not work for matrix polynomials of higher degree. It is therefore often convenient to work with linearizations instead of the original matrix polynomial, and to derive results about the original matrix polynomial using the properties shared between it and its linearization.

Depending on the problem, some linearizations will more or less effective than others, so it is useful to have a large family available. For example, it is desirable to have linearizations that are easily constructible. To ensure this, we will be considering companion forms, that is, $nk \times nk$ strong linearizations $L_P(\lambda) = \lambda L_1 - L_0$ of matrix polynomials $P(\lambda)$ of degree k such that each $n \times n$ block of L_1 and L_0 is either 0_n, $\pm I_n$, or $\pm A_i$, for $i = 0, 1, ..., k$, when L_1 and L_0 are viewed as $k \times k$ block matrices (6). For each matrix polynomial $P(\lambda)$, many different linearizations can be constructed but, in practice, those sharing the structure of $P(\lambda)$ are the most convenient from the theoretical and computational point of view, since the structure of $P(\lambda)$ often implies some symmetries in its spectrum, which are meaningful in physical applications and that can be destroyed by numerical methods when the structure is ignored (23). For example, if a matrix polynomial is real symmetric or Hermitian, its spectrum is symmetric with respect to the real axis, and the sets of left and right eigenvectors coincide. Thus, it is important to construct linearizations that reflect the structure of the original problem. In the literature (2; 4; 6; 12; 13; 19; 20) different kinds of structured linearizations have been considered: palindromic, symmetric, skew-symmetric, alternating, etc. We will be constructing linearizations that are symmetric when the matrix polynomial is that are also companion forms. Some examples of symmetric linearizations for symmetric matrix polynomials can be found in (1; 2; 12; 14; 15; 16; 18; 24); however, not all of these are companion forms.

From the numerical point of view, it is not enough to have linearizations that preserve the structure of the matrix polynomials. In any computational problem it is important to take into account its conditioning, i.e. its sensitivity to perturbations. It is known that different linearizations for a given polynomial eigenvalue problem can have very different conditioning (11; 22; 23). This implies that numerical methods may produce quite different results for different linearizations. Therefore, it is convenient to have available a large family of structured linearizations that can be constructed easily and from which a linearization as well-conditioned as the original problem can be chosen. To make the linearizations as useful as possible, we will attempt to find the largest possible family of companion forms that are symmetric when the original matrix polynomial is.

1.2 History of Fiedler Pencils and $\mathbb{DL}(P)$

Examples of companion forms had been found prior to 1985, (17), but there were only a few examples, and they did not preserve any sort of structure. There was also no guarantee that they would preserve the conditioning of the original matrix polynomial. As a first step towards fixing these problems, in 2003 M. Fiedler constructed a family of companion forms for regular matrix polynomials (a matrix polynomial $P(\lambda)$ is said to be *regular* if $\det P(\lambda) \neq 0$ for almost all λ) now known as Fiedler pencils (7).

Definition 4

$$M_0 := \left[\begin{array}{c|c} I_{(k-1)n} & 0 \\ \hline 0 & -A_0 \end{array}\right], \quad M_{-k} := \left[\begin{array}{c|c} A_k & 0 \\ \hline 0 & I_{(k-1)n} \end{array}\right],$$

and

$$M_i := \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & -A_i & I_n & 0 \\ 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right], \quad i = 1, \ldots, k-1. \tag{1.2}$$

The matrices M_i in (1.2) are always invertible and their inverses are given by

$$M_{-i} := M_i^{-1} = \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 \\ 0 & I_n & A_i & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right].$$

The matrices M_0 and M_{-k} are invertible if and only if A_0 and A_k, respectively, are.

Definition 5 *Let $i = (i_1, i_2, \ldots, i_n)$ be a permutation of $(1, 2, \ldots, n)$. Then the matrix polynomial*

$$\lambda M_0 - M_{i_1} M_{i_2} \ldots M_{i_n}$$

with the M_i as defined above is called the Fiedler pencil *of the matrix polynomial $P(\lambda)$ with respect to i.*

Fiedler pencils are easy to construct and retain many of the properties of the original matrices; however, they are a (relatively) small family of linearizations with only a $n!$ elements, and still do not preserve structure. For example, there are no linearizations in the family of Fiedler pencils that are guaranteed to be symmetric when $P(\lambda)$ is. The construction of Fiedler pencils led to the construction of much larger and more useful families of companion forms, starting with the discovery of generalized Fiedler pencils in 2004 by E. N. Antoniou and S. Vologiannidis (1; 2).

Definition 6 *(1) If I is a permutation of a set of indices, $I = \{i_0, i_1, \ldots, i_n\}$ we will let $M_I = M_{i_1} M_{i_2} \ldots M_{i_n}$. Let $\{C_0, C_1\}$ be a partition of $\{0, 1, \ldots k\}$ and let I_0 and I_1 be permutations of C_0 and $-C_1$. Then the pencil*

$$K(\lambda) = \lambda M_{I_0} - M_{I_1}$$

is the generalized Fiedler pencil *with respect to (I_0, I_1) with the M_i as defined above.*

In (1; 2) it was shown that there exists a companion form $L(\lambda)$ for $P(\lambda)$ that is symmetric when $P(\lambda)$ is, as well as a linearization that is alternating when $P(\lambda)$ is (a matrix polynomial is called *alternating* if $A_i^T = (-1)^i A_i$ for all i). However, there are still not very many structure preserving linearizations in this family. Generalized Fiedler pencils give many options for trying to match conditioning, or an option to preserve structure, but in general cannot do both. Generalized Fiedler pencils are also only companion forms for regular matrix polynomials, limiting their use. In 2011 E. N. Antoniou and S. Vologiannidis extended the family again, this time to Fiedler pencils with repetition (24).

Definition 7 *(24) We say that a tuple* $t = (i_1, i_2, \ldots, i_r)$ *with entries from* $(0, 1, \ldots, k)$ *satisfies the* SIP *if for every pair of indices* $i_a,\ i_b \in t$ *with* $i_a = i_b$*, and* $0 \leq a < b \leq r$ *there exists at least one* c *such that* $i_c = i_a + 1$ *and* $a < c < b$*.*

Definition 8 *(24) Let* $P(\lambda)$ *be a matrix polynomial of degree* k *with* A_0 *and* A_k *non-singular. Let* $0 \leq h \leq k-1$*, let* q *be a permutation of* $\{0, 1, \ldots, h\}$ *and* m *be a permutation of* $\{-k, -k+1, \ldots, -h-1\}$*. Let* l_q *and* r_q *be tuples with entries from* $(0, 1, \ldots, h-1)\}$ *such that* (l_q, q, r_q) *satisfies the SIP. Let* l_m *and* r_m *be tuples with entries in* $(-k, -k+1, \ldots, -h-2)$ *such that* (l_m, m, r_m) *satisfies the SIP. Then the pencil*

$$L(\lambda) = \lambda M_{l_m} M_{l_q} M_m M_{r_q} M_{r_m} - M_{l_m} M_{l_q} M_q M_{r_q} M_{r_m}$$

is a Fiedler pencil with repetition *(FPR) associated with* $P(\lambda)$*.*

It is worth noting that the condition that A_0 and A_k are non-singular is only needed for certain FPRs, so if we restrict ourselves to a subset of all possible FPRs (discussed later) we can avoid the non-singularity conditions that appear above and in other literature between 2004 and 2011. The main accomplishments of this generalization were this reduction of constraints on the non-singularity conditions, and finding more linearizations that are structure preserving, specifically symmetric linearizations for symmetric matrix polynomials. In (24) some symmetric linearizations in the family of FPR are found; however, not all of them are listed in the literature. In this paper we will find necessary and sufficient conditions for a FPR to be a symmetric linearization, expanding on the previous list.

In 2006 it was shown that it is possible to create vector spaces of pencils, almost all of which are linearizations for a given regular matrix polynomial, specifically $\mathbb{L}_1(P)$, $\mathbb{L}_2(P)$, and$\mathbb{DL}(P)$ (18).

Definition 9 *We will first introduce the notation*

$$\mathcal{V}_P = \{v \otimes P(\lambda) : v \in \mathbb{F}^k$$

We then define

$$\mathbb{L}_1(P) = \{L(\lambda) = \lambda X + Y :\ X,\ Y \in \mathbb{F}^{nk \times nk}, L(\lambda) \cdot (\Lambda \otimes I_n) \in \mathcal{V}_P\}$$

where Λ *denotes the vector* $[\lambda^{k-1},\ \lambda^{k-1},\ \ldots \lambda,\ 1]^T$*.*

Similarly, define

$$\mathcal{W}_P = \{w^T \otimes P(\lambda) : w \in \mathbb{F}^k\}$$

and

$$\mathbb{L}_2 = \{\{L(\lambda) = \lambda X + Y : \ X,\ Y \in \mathbb{F}^{nk\times nk},\ (\Lambda^T \otimes I_n) \cdot L(\lambda) \in \mathcal{W}_P\}$$

We then define $\mathbb{DL}_P = \mathbb{L}_1 \cap \mathbb{L}_2$.

These are significant because from linearizations in $\mathbb{L}_1(P)$ one can deduce the left eigenvectors of $P(\lambda)$ and from $\mathbb{L}_2(P)$ one can deduce the right eigenvectors of $P(\lambda)$ so from linearizations in $\mathbb{DL}(P)$ one can deduce both the left and right eigenvectors for $P(\lambda)$. Furthermore, a pencil in $\mathbb{DL}(P)$ is symmetric whenever $P(\lambda)$ is. We note that in (24) it is shown that, if the matrix coefficients A_0 and A_k of the matrix polynomial $P(\lambda)$ are nonsingular, the family of FPR includes k symmetric linearizations presented in (14; 15; 16), which form a basis of the k-dimensional vector space of symmetric pencils $\mathbb{DL}(P)$ studied in (18; 12). Note that, though any pencil in $\mathbb{DL}(P)$ is symmetric when $P(\lambda)$ is, it is not necessarily a linearization of $P(\lambda)$. Moreover, any symmetric pencil in $\mathbb{L}_1(P)$ is in $\mathbb{DL}(P)$.

Although symmetric linearizations for symmetric matrix polynomials have been found, from example 8 in (24) the family of symmetric strong linearizations among the FPR includes linearizations that had not appeared in the literature before. In particular these linearizations are not in $\mathbb{DL}(P)$. While in (24) only a few examples were constructed, in this paper, we characterize all the pencils in the family of Fiedler pencils with repetition which are symmetric when the associated matrix polynomial $P(\lambda)$ is. Though not every pencil in this family is a linearization of $P(\lambda)$, we give the conditions under which they are strong linearizations. In particular, when A_0 and A_k are both nonsingular, the family of symmetric strong FPR linearizations in the FPR that we construct extends the basis of the space $\mathbb{DL}(P)$ significantly, as Example 59 shows for $k = 4$. Notice that in this case we get 10 linearly independent linearizations. It remains an open problem to determine the dimension of the vector space of symmetric pencils generated by our FPR linearizations, though it is clear from Theorem 57 that this dimension is always greater than the degree k of the matrix polynomial. Notice that, applying this theorem for $k \geq 3$ taking $\mathbf{w} = (1:2,0)$, $\mathbf{r}_w = (1)$ and $\mathbf{t}_w = \emptyset$ ($\mathbf{z}$ can be any admissible tuple), we get an example of a strong linearization that is not in $\mathbb{L}_1$ and, therefore, not in $\mathbb{DL}(P)$. This follows since the matrix coefficient of the term of degree 0 of this pencil contains at least one identity block and the matrix coefficient of the term of degree 1 does not contain any block equal to $-I_n$.

1.3 Structure of the paper

In section 2 we will introduce tuples and prove some basic theorems relating to them. We will introduce terminology that is used throughout the paper, and prove various results that we will use as tools later. It is an overview of known properties that will be useful to prove original results later on. In section 3 we will start to study properties of symmetric tuples. In particular, we will characterize the tuples that are relevant to creating symmetric linearizations that are FPRs. We will do this by first defining properties of tuples that will determine if an FPR can be constructed from them. We will then find all tuples with these conditions that are symmetric, such that the corresponding FPR will be symmetric. Finally, we will define a canonical form on our tuples, so that after reduction to the canonical form no two tuples will correspond to the same FPR. The results in this section are all original work. In section 4 we will go over known properties of FPR, and demonstrate the connection between FPR and the tuples discussed in section 3. In section 5 we will relate our work on symmetric tuples to symmetric FPR and prove our main result, a complete description of FPR that are symmetric (Hermitian) strong linearizations when the matrix polynomial $P(\lambda)$ is.

Chapter 2

Index Tuples

We call an *index tuple* any ordered tuple whose entries are integers.

In this section we introduce some definitions and results for index tuples. In particular, we define an equivalence relation in the set of index tuples and give a canonical form under this equivalence relation. We also give some notation that will be used throughout the paper.

2.1 General definitions and notation

For the purposes of this paper, it is important to distinguish between index tuples in which the entries are repeated or not. This justifies the following definition.

Definition 10 *Let* $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ *be an index tuple. We say that* $\mathbf{t}$ *is* simple *if* $i_j \neq i_l$ *for all* $j, l \in \{1, 2, \ldots, r\}$, $j \neq l$.

If i, j are integers such that $j \geq i$, we denote by $(i : j)$ the tuple $(i, i+1, i+2, \ldots, j)$. If $j < i$, $(i : j)$ denotes the empty tuple. We will refer to the simple index tuple $(i : j)$, $j \geq i$, consisting of consecutive integers, as a *string*.

If i, j are integers such that $j \leq i$, we denote by $\left(i :_{\downarrow} j\right)$ the tuple $(i, i-1, i-2, \ldots, j)$. If $j > i$, $\left(i :_{\downarrow} j\right)$ denotes the empty tuple.

Definition 11 *Let* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *be two index tuples. We say that* $\mathbf{t}_1$ *is a* subtuple *of* $\mathbf{t}_2$ *if* $\mathbf{t}_1 = \mathbf{t}_2$ *or if* $\mathbf{t}_1$ *can be obtained from* $\mathbf{t}_2$ *by deleting some indices in* $\mathbf{t}_2$. *If* $i_1, \ldots, i_r$ *are distinct indices, we call the subtuple of* $\mathbf{t}_1$ *with indices from* $\{i_1, \ldots, i_r\}$ *the subtuple of* $\mathbf{t}_1$ *obtained by deleting from* $\mathbf{t}_1$ *all indices distinct from* $i_1, \ldots, i_r$.

Example 12 *Let* $\mathbf{t} = (1,2,1,3,2,3)$ *be viewed as a tuple with indices from* $\{1,2,3,4\}$. *The subtuple of* $\mathbf{t}$ *with indices from* $\{1,2\}$ *is* $(1,2,1,2)$*; the subtuple of* $\mathbf{t}$ *with indices from* $\{3,4\}$ *is* $(3,3)$.

Note that in a subtuple of an index tuple, the indices keep their original relative positions, that is, the order of the indices in the subtuple is not altered with respect to the order of those indices in the original tuple.

Given an index tuple $\mathbf{t} = (i_1, \ldots, i_r)$ and an integer a, we denote by $a + \mathbf{t}$ the index tuple $(a + i_1, \ldots, a + i_r)$.

Given an index tuple $\mathbf{t}$, we call the number of elements in $\mathbf{t}$ the *length* of $\mathbf{t}$ and denote it by $|\mathbf{t}|$. We denote by $\mathbf{t}[j]$ the tuple obtained from $\mathbf{t}$ by deleting the last j elements (counting from the right).

Definition 13 *Let* $\mathbf{t} = (a : b)$ *be a string and* $l = |\mathbf{t}|$. *If* $l > 1$, *we call the* reverse-complement *of* $\mathbf{t}$ *the index tuple* $\mathbf{t}_{rev_c} := (\mathbf{t}[1], \ldots, \mathbf{t}[l-1])$. *If* $l = 1$, *the reverse-complement of* $\mathbf{t}$ *is empty.*

Example 14 *The reverse-complement of* $\mathbf{t} = (0 : 6)$ *is* $\mathbf{t}_{rev_c} = (0 : 5, 0 : 4, 0 : 3, 0 : 2, 0 : 1, 0)$*; the reverse-complement of* $\mathbf{t} = (0)$ *is empty.*

Definition 15 *Given an index tuple* $\mathbf{t} = (i_1, \ldots, i_r)$, *we define the* reversal tuple *of* $\mathbf{t}$ *as* $\mathrm{rev}(\mathbf{t}) := (i_r, \ldots, i_1)$.

Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two index tuples. Some immediate properties of the reversal operation are:

- $rev(rev(\mathbf{t}_1)) = \mathbf{t}_1$,
- $rev(\mathbf{t}_1, \mathbf{t}_2) = (rev(\mathbf{t}_2), rev(\mathbf{t}_1))$.

2.2 Equivalence of tuples

We define an equivalence relation in the set of index tuples with indices from a given set of either nonnegative or negative integers.

Definition 16 *We say that two nonnegative indices* i, j commute *if* $|i - j| \neq 1$.

Definition 17 *Let* $\mathbf{t}$ *and* $\mathbf{t}'$ *be two index tuples of nonnegative integers. We say that* $\mathbf{t}$ *is obtained from* $\mathbf{t}'$ *by a* transposition *if* $\mathbf{t}$ *is obtained from* $\mathbf{t}'$ *by interchanging two commuting indices in adjacent positions.*

Definition 18 *Given two index tuples* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *of nonnegative integers, we say that* $\mathbf{t}_1$ *is* equivalent *to* $\mathbf{t}_2$ *if* $\mathbf{t}_2$ *can be obtained from* $\mathbf{t}_1$ *by a sequence of transpositions. If* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *are index tuples of negative indices, we say that* $\mathbf{t}_1$ *is equivalent to* $\mathbf{t}_2$ *if* $-\mathbf{t}_1$ *is equivalent to* $-\mathbf{t}_2$*. If* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *are equivalent index tuples, we write* $\mathbf{t}_1 \sim \mathbf{t}_2$*,*

Note that the relation $\sim$ is an equivalence relation.

Example 19 *The index tuples* $\mathbf{t}_1 = (2,5,3,1,4)$ *and* $\mathbf{t}_2 = (5,2,3,4,1)$ *are equivalent.*

Remark 20 *Note that if* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *are equivalent tuples with indices from* $\{i, i+1\}$*, where* i *is a nonnegative integer, then* $\mathbf{t}_1 = \mathbf{t}_2$*.*

Observe that, if j is an integer and $\mathbf{t}_1$ and $\mathbf{t}_2$ are equivalent index tuples, then so are $j + \mathbf{t}_1$ and $j + \mathbf{t}_2$.

The next proposition will be very useful in the proofs of our results.

Proposition 21 *Let* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *be two index tuples with indices from a set* S *of nonnegative (resp. negative) integers. Then,* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *are equivalent if and only if, for each* $i \in S$*, the subtuples of* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *with indices from* $\{i, i+1\}$ *are the same.*

Proof. If $\mathbf{t}_1$ and $\mathbf{t}_2$ are equivalent then they contain the same indices with the same multiplicities, and, since i and $i+1$ do not commute, the stated subtuples are the same. For the converse, assume that $\mathbf{t}_1$ and $\mathbf{t}_2$ are not equivalent. If $\mathbf{t}_1$ and $\mathbf{t}_2$ do not have the same indices, clearly for some $i \in S$ the subtuples with indices from $\{i, i+1\}$ are distinct. Now suppose that $\mathbf{t}_1$ and $\mathbf{t}_2$ have the same indices. Let k be the first position (starting from the left) in which $\mathbf{t}_1$ and $\mathbf{t}_2$ differ and no transposition applied to the indices of $\mathbf{t}_2$ to the right of position $k-1$ can transform the index in position k into the corresponding index in $\mathbf{t}_1$, say i. Since, by applying transpositions on $\mathbf{t}_2$, we cannot find an equivalent tuple with i in position k (and the elements in the positions before k are equal in both tuples) this means that $i-1$ or $i+1$ appears to the right of position $k-1$ and to the left of the first i after position k in $\mathbf{t}_2$. But this implies that either the subtuples of $\mathbf{t}_1$ and $\mathbf{t}_2$ with indices from $\{i, i-1\}$ or the subtuples of $\mathbf{t}_1$ and $\mathbf{t}_2$ with indices from $\{i, i+1\}$ are different. ■

The next example illustrates the application of Proposition 21.

Example 22 *Consider the tuples* $\mathbf{t}_1 = (1,5,4,2)$ *and* $\mathbf{t}_2 = (5,1,2,4)$ *with indices from* $S = \{1,2,4,5\}$*. For each* $i \in S$*, the subtuples of* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *with indices* *from $\{i, i+1\}$ coincide and are given by $(1,2)$ if $i = 1$, (2) if $i = 2$, $(5,4)$ if $i = 4$, and (5) if $i = 5$. Thus, by Proposition 21, $\mathbf{t}_1$ and $\mathbf{t}_2$ are equivalent. Now consider the tuples $\mathbf{t}_1 = (5,6,25)$ and $\mathbf{t}_2 = (5,6,30)$ with indices from $S = \{5,6,25,30\}$. Clearly, the subtuples of $\mathbf{t}_1$ and $\mathbf{t}_2$ with indices from $\{i, i+1\}$, when $i = 25$, do not coincide. Thus, by Proposition 21, $\mathbf{t}_1$ and $\mathbf{t}_2$ are not equivalent.*

The next result is an easy consequence of the previous proposition and will be used in the proofs of our results.

Lemma 23 *Let $\mathbf{q}$ be a permutation of $\{0,1,\ldots,h\}$, $h \geq 0$, and $\mathbf{l}_q, \mathbf{r}_q, \mathbf{l}'_q, \mathbf{r}'_q$ be tuples with indices from $\{0,1,\ldots,h-1\}$. Then, $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (\mathbf{l}'_q, \mathbf{q}, \mathbf{r}'_q)$ if and only if $\mathbf{l}_q \sim \mathbf{l}'_q$ and $\mathbf{r}_q \sim \mathbf{r}'_q$.*

Proof. Clearly, if $\mathbf{l}_q \sim \mathbf{l}'_q$ and $\mathbf{r}_q \sim \mathbf{r}'_q$ then $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (\mathbf{l}'_q, \mathbf{q}, \mathbf{r}'_q)$. Now we prove the converse. Suppose that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (\mathbf{l}'_q, \mathbf{q}, \mathbf{r}'_q)$. We prove that $\mathbf{r}_q \sim \mathbf{r}'_q$. The proof of $\mathbf{l}_q \sim \mathbf{l}'_q$ is similar. By Proposition 21, it is enough to show that, for any index $i \in \{0,\ldots,h-1\}$, the subtuples of $\mathbf{r}_q$ and $\mathbf{r}'_q$ with indices from $\{i, i+1\}$ are the same. First we prove that $\mathbf{r}_q$ and $\mathbf{r}'_q$ have precisely the same indices.

In order to get a contradiction, assume that $i \leq h-1$ is the largest index such that the subtuples of $\mathbf{r}_q$ and $\mathbf{r}'_q$ with indices from $\{i\}$ have different lengths. Let m denote the number of indices equal to $i+1$ in $\mathbf{r}_q$ and $\mathbf{r}'_q$ (which can be 0). By Proposition 21, the subtuples of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}'_q, \mathbf{q}, \mathbf{r}'_q)$ with indices from $\{i, i+1\}$ are the same, which gives a contradiction as the number of i's occurring to the right of the $(m+1)$th index equal to $i+1$, counting from the right, is different in both tuples.

Thus, we conclude that $\mathbf{r}_q$ and $\mathbf{r}'_q$ have precisely the same indices. Since, by Proposition 21, for each $i < h$, the subtuples of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}'_q, \mathbf{q}, \mathbf{r}'_q)$ with indices from $\{i, i+1\}$ are the same, also the corresponding subtuples of $\mathbf{r}_q$ and $\mathbf{r}'_q$ are the same. Again by Proposition 21, the claim follows. ■

We now extend the definition of commuting indices to index tuples.

Definition 24 *Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two index tuples of nonnegative (resp. negative) integers. We say that $\mathbf{t}_1$ and $\mathbf{t}_2$* commute *if every index in $\mathbf{t}_1$ commutes with every index in $\mathbf{t}_2$.*

Note that, if $\mathbf{t}_1$ and $\mathbf{t}_2$ commute, then, for every index i in $\mathbf{t}_1$, $i-1$ and $i+1$ are not in $\mathbf{t}_2$. In particular, if $\mathbf{t}_1$ and $\mathbf{t}_2$ commute then $(\mathbf{t}_1, \mathbf{t}_2) \sim (\mathbf{t}_2, \mathbf{t}_1)$. Also, if $\mathbf{t}'_1$ and $\mathbf{t}'_2$ are subtuples of the commuting tuples $\mathbf{t}_1$ and $\mathbf{t}_2$, then $\mathbf{t}'_1$ and $\mathbf{t}'_2$ commute.

2.3 Successor Infix Property and column standard form

In this paper we are interested in index tuples satisfying the property called SIP that we define below. In the case of tuples of nonnegative indices satisfying this property, we also give a canonical form under the equivalence relation defined in the previous section. Expressing the index tuples satisfying the SIP in this canonical form is an important tool for proving our main results.

Definition 25 *(24) Let* $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ *be an index tuple of nonnegative (resp. negative) indices. Then,* $\mathbf{t}$ *is said to satisfy the* Successor Infix Property (SIP) *if for every pair of indices* $i_a, i_b \in \mathbf{t}$*, with* $1 \leq a < b \leq r$*, satisfying* $i_a = i_b$*, there exists at least one index* $i_c = i_a + 1$ *with* $a < c < b$*.*

Remark 26 *We observe that the SIP is invariant under equivalence. Moreover, any subtuple consisting of adjacent indices from an index tuple satisfying the SIP also satisfies the SIP. Additionally, if a tuple* $\mathbf{t}$ *satisfies the SIP, so does* $rev(\mathbf{t})$ *and* $a + \mathbf{t}$ *for any integer* a*.*

We now give a canonical form under equivalence for a tuple of nonnegative integers satisfying the SIP. Note that Definition 25 was presented for arbitrary tuples, not necessarily with nonnegative indices, as for the definition of the symmetric linearizations we will need it in that general form.

Definition 27 *(24) Let* $\mathbf{t}$ *be an index tuple with indices from* $\{0, 1, \ldots, h\}$, $h \geq 0$*. Then* $\mathbf{t}$ *is said to be in* column standard form *if*

$$\mathbf{t} = (a_s : b_s, a_{s-1} : b_{s-1}, \ldots, a_2 : b_2, a_1 : b_1),$$

with $h \geq b_s > b_{s-1} > \cdots > b_2 > b_1 \geq 0$ *and* $0 \leq a_j \leq b_j$*, for all* $j = 1, \ldots, s$*. We call each subtuple of consecutive integers* $(a_i : b_i)$ *a* string *in* $\mathbf{t}$*.*

The connection between the column standard form of an index tuple and the SIP is shown in the following result.

Lemma 28 *(24) Let* $\mathbf{t} = (i_1, \ldots, i_r)$ *be an index tuple with indices from* $\{0, 1, \ldots, h\}$*,* $h \geq 0$*. Then* $\mathbf{t}$ *satisfies the SIP if and only if* $\mathbf{t}$ *is equivalent to a tuple in column standard form.*

Taking into account Proposition 21, it follows that two tuples in column standard form are equivalent if and only if they coincide. We then have the following definition.

Definition 29 *The unique index tuple in column standard form equivalent to an index tuple* $\mathbf{t}$ *satisfying the SIP is called the* column standard form of t *and is denoted by* $csf(\mathbf{t})$.

Note that, in particular, if $\mathbf{t}$ is simple, then $\mathbf{t}$ satisfies the SIP and, therefore, $\mathbf{t}$ is equivalent to a tuple in column standard form. In this case, if $\mathbf{t}$ is a permutation of $\{0, 1, \ldots, h\}$, the column standard form of $\mathbf{t}$ has the form

$$csf(\mathbf{t}) = (t_w + 1 : h, t_{w-1} + 1 : t_w, \ldots, t_2 + 1 : t_3, t_1 + 1 : t_2, 0 : t_1)$$

for some positive integers $0 \leq t_1 < t_2 < \ldots < t_w < h$.

Chapter 3

Symmetric Index Tuples

In this section we consider index tuples that are symmetric.

Definition 30 *We say that an index tuple* $\mathbf{t}$ *of nonnegative (resp. negative) indices is* symmetric *if* $\mathbf{t} \sim rev(\mathbf{t})$.

Observe that any tuple equivalent to a symmetric tuple is also symmetric.

We are interested in symmetric tuples of the form $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfying the SIP, where $\mathbf{q}$ is a permutation of $\{0, 1, \ldots, h\}$, $\mathbf{l}_q$ and $\mathbf{r}_q$ are tuples (possibly not simple) with indices from $\{0, 1, \ldots, h-1\}$, and $(\mathbf{l}_q, \mathbf{r}_q)$ is also symmetric. We characterize the symmetric tuples of this kind and give a new canonical form under equivalence for them. The canonical form we present will be used in the construction of the symmetric linearizations.

3.1 The S and the SS properties

Here we introduce some properties of symmetric tuples that will be very useful in proving our results. We focus on nonnegative tuples but all the results in this section can be extended to tuples of negative indices as well.

Definition 31 *Let* $\mathbf{t}$ *be a tuple with indices from* $\{0, 1, ..., h\}$, $h \geq 0$. *We say that* $\mathbf{t}$ *has the* S property *if, for every index* $i \in \mathbf{t}$ *with* $i < h$, *the subtuple of* $\mathbf{t}$ *with indices from* $\{i, i+1\}$ *is symmetric. In particular, if for every index* $i \in \mathbf{t}$ *with* $i < h$ *such that* $i + 1 \in \mathbf{t}$, *the subtuple of* $\mathbf{t}$ *with indices from* $\{i, i+1\}$ *is of the form* $(i, i+1, i, i+1, ..., i+1, i)$ *or* $(i+1, i, i+1, ..., i, i+1)$, *we say that* $\mathbf{t}$ *has the* SS property.

Lemma 32 *Let* $\mathbf{t}$ *be a tuple with indices from* $\{0, 1, \ldots, h\}$*,* $h \geq 0$*. Then,* $\mathbf{t}$ *is symmetric if and only if* $\mathbf{t}$ *has the S property.*

Proof. If $\mathbf{t}$ is symmetric, then it is clear that $\mathbf{t}$ has the S property. Now assume that $\mathbf{t}$ is not symmetric in order to see that $\mathbf{t}$ does not satisfy the S property. Since $\mathbf{t}$ and $rev(\mathbf{t})$ are not equivalent, by Proposition 21, there is $i \in \mathbf{t}$ such that the subtuples of $\mathbf{t}$ and $rev(\mathbf{t})$ with indices from $\{i, i+1\}$ are distinct. Thus, the subtuple of $\mathbf{t}$ with indices from $\{i, i+1\}$ is not symmetric, which implies the result. ■

In order to characterize the index tuples $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ which are symmetric and such that $(\mathbf{l}_q, \mathbf{r}_q)$ is also symmetric, we start by considering the case when $\mathbf{l}_q$ and $\mathbf{r}_q$ are disjoint tuples (that is, have no common indices).

Lemma 33 *Let* $\mathbf{q}$ *be a permutation of* $\{0, 1, \ldots, h\}$*,* $h \geq 0$*, and let* $\mathbf{l}_q, \mathbf{r}_q$ *be disjoint tuples with indices from* $\{0, 1, ..., h-1\}$ *such that* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ *or* $(\mathbf{l}_q, \mathbf{r}_q)$ *is symmetric. Then,* $\mathbf{l}_q$ *and* $\mathbf{r}_q$ *commute.*

Proof. We observe that there is no index i such that either $i \in \mathbf{l}_q$ and $i+1 \in \mathbf{r}_q$ or $i \in \mathbf{r}_q$ and $i+1 \in \mathbf{l}_q$, as, otherwise, the subtuple of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ (or $(\mathbf{l}_q, \mathbf{r}_q)$) with indices from $\{i, i+1\}$ would not be symmetric, (as its first and last elements would be different), a contradiction by Lemma 32. ■

Next we characterize, in terms of the SS property, the index tuples $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfying the SIP, with $\mathbf{l}_q$ and $\mathbf{r}_q$ disjoint and such that both $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_q, \mathbf{r}_q)$ are symmetric. Note that if $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric and $\mathbf{l}_q$ and $\mathbf{r}_q$ are disjoint, from Lemmas 32 and 33, $\mathbf{l}_q$ and $\mathbf{r}_q$ are symmetric as well.

Lemma 34 *Let* $\mathbf{q}$ *be a permutation of* $\{0, 1, \ldots, h\}$*,* $h \geq 0$*, and let* $\mathbf{l}_q, \mathbf{r}_q$ *be disjoint tuples with indices from* $\{0, 1, ..., h-1\}$ *such that* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ *satisfies the SIP. Then,* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ *and* $(\mathbf{l}_q, \mathbf{r}_q)$ *are both symmetric if and only if* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ *has the SS property.*

Proof. Assume that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ has the SS property, which implies that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ has the S property. Then, by Lemma 32 and taking into account that, for every $i \in \{0, 1, \ldots, h-1\}$, the subtuple of $\mathbf{q}$ with indices from $\{i, i+1\}$ is of the form $(i, i+1)$ or $(i+1, i)$, the result follows.

Assume now that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_q, \mathbf{r}_q)$ are both symmetric. Let $i \in \{0, 1, \ldots, h-1\}$. By the SIP, the subtuple $\mathbf{j}$ of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ with indices from $\{i, i+1\}$ cannot have two adjacent i's. We next show that $\mathbf{j}$ cannot have two adjacent elements equal to $i+1$ either. Assume it does. Since $\mathbf{q}$ only contains one index $i+1$ and $\mathbf{l}_q$ and $\mathbf{r}_q$ are disjoint, we have either $i+1 \in \mathbf{l}_q$ or $i+1 \in \mathbf{r}_q$. Suppose that $i+1 \in \mathbf{r}_q$ (which implies that $i+1 \notin \mathbf{l}_q$). The argument is analogous if $i+1 \in \mathbf{l}_q$. By Lemma 33, $i \notin \mathbf{l}_q$. Let p be the smallest positive integer such that the entries in positions p and $p+1$ in $\mathbf{j}$ are $i+1$. Note that $p \geq 2$, since $\mathbf{q}$ contains one i and one $i+1$. Also, the entry in position $p-1$ in the subtuple of $\mathbf{r}_q$ with indices from $\{i, i+1\}$ (which is the entry in position $p+1$ in the subtuple $\mathbf{j}$) is $i+1$. Because $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric and $i, i+1 \notin \mathbf{l}_q$, by Lemma 32, the subtuple of $\mathbf{r}_q$ with indices from $\{i, i+1\}$ is symmetric. Thus, the $(p-1)$th element counting from right to left in $\mathbf{r}_q$ (and, therefore, in $\mathbf{j}$) is $i+1$. Since, by Lemma 32, the subtuple $\mathbf{j}$ is also symmetric, we would get that the entry in position $p-1$ in $\mathbf{j}$ is $i+1$, a contradiction. Thus, we have shown that, in the subtuple $\mathbf{j}$, the indices i and $i+1$ alternate. Since, by Lemma 32, the subtuple $\mathbf{j}$ is symmetric, the first and last entry of $\mathbf{j}$ are equal and the result follows. ■

3.2 Admissible Tuples

Here we introduce the concept of admissible tuple which will allow us to find a new canonical form under equivalence for symmetric tuples of the form $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$. This canonical form will be very useful in the construction of symmetric linearizations.

Definition 35 *Let $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, h\}$, $h \geq 0$. We say that $\mathbf{q}$ is an* admissible tuple *relative to $\{0, 1, \ldots, h\}$ if the sequence of the lengths of the strings in $csf(\mathbf{q})$ is of the form $(2, ..., 2, l+1)$, where $l \geq 0$. We call l the index of $\mathbf{q}$.*

From now on, in order to make our statements clearer, we will associate to an arbitrary permutation of $\{0, 1, \ldots, h\}$ the letter $\mathbf{q}$ and to an admissible tuple the letter $\mathbf{w}$.

Example 36 *Here we give some examples of admissible index tuples.*

- $\mathbf{w}_1 = (6:7, 4:5, 0:3)$ *is an admissible tuple with index* 3 *relative to* $\{0, \ldots, 7\}$.
- $\mathbf{w}_2 = (5:6, 3:4, 1:2, 0)$ *is an admissible tuple with index* 0 *relative to* $\{0, \ldots, 6\}$.

Note that if $\mathbf{w}$ is an admissible tuple with index l relative to $\{0, 1, ..., h\}$, then h and l have the same parity.

In the next definition we construct an index tuple that, when appended to an admissible tuple, produces a symmetric index tuple. We use the notation for the reverse-complement of a tuple introduced in Definition 13.

Definition 37 *(Symmetric complement) Let* $\mathbf{w}$ *be an admissible tuple with index* l *relative to* $\{0,1,\ldots,h\}$, $h \geq 0$. *We call the* symmetric complement of $\mathbf{w}$ *the tuple* $\mathbf{r}_w$ *defined as follows:*

- $\mathbf{r}_w = (h-1, h-3, ..., l+3, l+1, (0:l)_{rev_c})$, *if* $l \geq 1$,
- $\mathbf{r}_w = (h-1, h-3, ..., 1)$, *if* $l = 0$.

Example 38 *The symmetric complements of the tuples* $\mathbf{w}_1$ *and* $\mathbf{w}_2$ *given in Example 36 are*

$$\mathbf{r}_{w_1} = (6,4,0:2,0:1,0) \qquad \textit{and} \qquad \mathbf{r}_{w_2} = (5,3,1),$$

respectively.

We next show that, if $\mathbf{w}$ is an admissible index tuple and $\mathbf{r}_w$ is the symmetric complement of $\mathbf{w}$, then $(\mathbf{w}, \mathbf{r}_w)$ is symmetric. We need the following auxiliary result.

Proposition 39 *The reverse-complement of the string* $\mathbf{t} = (0:l)$, $l \geq 1$, *is symmetric and satisfies the SIP.*

Proof. Since $\mathbf{t}_{rev_c}$ is in column standard form, by Lemma 28, it satisfies the SIP. The rest of the proof is by induction on l. If $l = 1$, the result holds trivially. Now suppose that $l > 1$. Let $\mathbf{r}_i = (0:i)$, $i = 0, \ldots, l-1$, so that $\mathbf{t}_{rev_c} = (\mathbf{r}_{l-1}, \ldots, \mathbf{r}_0)$. Note that $(0:l-1)_{rev_c} = (\mathbf{r}_{l-2}, \ldots, \mathbf{r}_0)$. By the induction hypothesis,

$$(\mathbf{r}_{l-2}, \ldots, \mathbf{r}_0) \sim rev(\mathbf{r}_{l-2}, \ldots, \mathbf{r}_0).$$

Then,

$$\begin{aligned} rev(\mathbf{t}_{rev_c}) &= (rev(\mathbf{r}_{l-2}, \ldots, \mathbf{r}_0), rev(\mathbf{r}_{l-1})) \\ &\sim (\mathbf{r}_{l-2}, \ldots, \mathbf{r}_0, l-1 :_{\downarrow} 0) \\ &\sim (\mathbf{r}_{l-2}, l-1, \mathbf{r}_{l-3}, l-2, \ldots, \mathbf{r}_0, 1, 0) = \mathbf{t}_{rev_c}, \end{aligned}$$

where the last equivalence follows from the commutativity relations for indices. ■

Lemma 40 *Let* $\mathbf{w}$ *be an admissible tuple with index* l *relative to* $\{0,1,\ldots,h\}$, $h \geq 0$. *Let* $\mathbf{r}_w$ *be the symmetric complement of* $\mathbf{w}$. *Then,* $(\mathbf{w}, \mathbf{r}_w)$ *is symmetric and satisfies the SIP. Moreover,* $\mathbf{r}_w$ *is symmetric.*

Proof. The fact that $(\mathbf{w}, \mathbf{r}_w)$ satisfies the SIP follows from the definition of $\mathbf{r}_w$ and Proposition 39. Also, by Proposition 39 and taking into account the commutativity relations for indices, it follows that the tuple $\mathbf{r}_w$ is symmetric.

Now we show that $(\mathbf{w}, \mathbf{r}_w)$ is symmetric. Assume that $csf(\mathbf{w}) = (B_s, \ldots, B_0)$, where B_i, $i = 0, \ldots, s$, are the strings of $csf(\mathbf{w})$. We prove the result by induction on s. If $s = 0$ the claim follows from Proposition 39 taking into account that $(\mathbf{w}, \mathbf{r}_w)$ is the reverse complement of $(0 : h+1)$. Now suppose that $s > 0$. Then, $\mathbf{w}' = (B_{s-1}, \ldots, B_0)$ is an admissible tuple. Let $\mathbf{r}_{w'}$ be the symmetric complement of $\mathbf{w}'$. Note that $B_s = (h-1 : h)$ and $\mathbf{r}_w \sim (\mathbf{r}_{w'}, h-1)$. Thus,

$$(\mathbf{w}, \mathbf{r}_w) \sim (h-1, h, \mathbf{w}', \mathbf{r}_{w'}, h-1)$$

So, we have

$$\begin{aligned} rev(\mathbf{w}, \mathbf{r}_w) &\sim (h-1, rev(\mathbf{w}', \mathbf{r}_{w'}), h, h-1) \\ &\sim (h-1, \mathbf{w}', \mathbf{r}_{w'}, h, h-1) \\ &\sim (h-1, h, \mathbf{w}', \mathbf{r}_{w'}, h-1) \\ &\sim (\mathbf{w}, \mathbf{r}_w), \end{aligned}$$

where the second equivalence follows from the induction hypothesis and the third equivalence follows because the largest index in $(\mathbf{w}', \mathbf{r}_{w'})$ is $h-2$ and, therefore, h commutes with any index in $(\mathbf{w}', \mathbf{r}_{w'})$. ■

Remark 41 *Note that, if* $\mathbf{w}$ *is an admissible tuple with indices from* $\{0, 1, ..., h\}$, $h < k$, *and* $\mathbf{r}_w$ *is the corresponding symmetric complement, then* $(-k + \mathbf{w}, -k + \mathbf{r}_w)$ *and* $-k + \mathbf{r}_w$ *are symmetric.*

3.3 Reduction to the Admissible Case

In this section we first prove that every symmetric index tuple of the form $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfying the SIP and such that $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric is equivalent to an index tuple of the form $(rev(\mathbf{t}), \mathbf{l}_q^*, \mathbf{q}, \mathbf{r}_q^*, \mathbf{t})$ with $\mathbf{l}_q^*$ and $\mathbf{r}_q^*$ disjoint. Then we show that $(\mathbf{l}_q^*, \mathbf{q}, \mathbf{r}_q^*)$ is equivalent to an index tuple of the form $(rev(\mathbf{t}'), \mathbf{w}, \mathbf{r}_w, \mathbf{t}')$, where $\mathbf{w}$ is an admissible tuple and $\mathbf{r}_w$ is the associated symmetric complement.

Lemma 42 *Let* $\mathbf{q}$ *be a permutation of* $\{0, 1, \ldots, h\}$, $h \geq 0$, *and* $\mathbf{l}_q, \mathbf{r}_q$ *be tuples with indices from* $\{0, 1, ..., h-1\}$ *such that* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ *satisfies the SIP. Suppose* *that* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ *and* $(\mathbf{l}_q, \mathbf{r}_q)$ *are symmetric. Then, there exist unique (up to equivalence) index tuples* $\mathbf{t}$, $\mathbf{l}_q^*$, $\mathbf{r}_q^*$, *with indices from* $\{0, \ldots, h-1\}$, *such that* $\mathbf{l}_q^*$ *and* $\mathbf{r}_q^*$ *are disjoint and*

$$(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (rev(\mathbf{t}), \mathbf{l}_q^*, \mathbf{q}, \mathbf{r}_q^*, \mathbf{t}). \tag{3.1}$$

Moreover,

$$\mathbf{l}_q \sim (rev(\mathbf{t}), \mathbf{l}_q^*) \qquad \mathbf{r}_q \sim (\mathbf{r}_q^*, \mathbf{t}), \tag{3.2}$$

and $(\mathbf{l}_q^*, \mathbf{q}, \mathbf{r}_q^*)$ *and* $(\mathbf{l}_q^*, \mathbf{r}_q^*)$ *are symmetric.*

Proof. Assume that $\mathbf{l}_q$ and $\mathbf{r}_q$ are not disjoint, otherwise the existence claim follows with $\mathbf{t} = \emptyset$, $\mathbf{l}_q^* = \mathbf{l}_q$, and $\mathbf{r}_q^* = \mathbf{r}_q$. Let $\mathbf{l}_q = (i_1, \mathbf{l}_q')$ for some index i_1 and some index tuple $\mathbf{l}_q'$. Then, because $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ is symmetric, we have $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (i_1, \mathbf{l}_q', \mathbf{j}, i_1)$, for some tuple $\mathbf{j}$. Therefore, if $i_1 \notin \mathbf{r}_q$, then $\mathbf{j} \sim (q', \mathbf{r}_q)$, where $\mathbf{q}'$ is the subtuple obtained from $\mathbf{q}$ by deleting the index i_1, and i_1 commutes with $\mathbf{r}_q$. Repeating this argument, we get that any index in $\mathbf{l}_q$ on the left of the first index in both $\mathbf{l}_q$ and $\mathbf{r}_q$, say j, should commute with j. Thus, since $\mathbf{l}_q$ and $\mathbf{r}_q$ are not disjoint, we can commute the indices in $\mathbf{l}_q$ in order to have in the first position on the left an index in both $\mathbf{l}_q$ and $\mathbf{r}_q$. So, assume that $i_1 \in \mathbf{r}_q$. Moreover, because $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ is symmetric, we have $\mathbf{r}_q \sim (\mathbf{r}_q', i_1)$ for some index tuple $\mathbf{r}_q'$. Thus,

$$(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (i_1, \mathbf{l}_q', \mathbf{q}, \mathbf{r}_q', \mathbf{i}_1).$$

Clearly, $(\mathbf{l}_q', \mathbf{q}, \mathbf{r}_q')$ and $(\mathbf{l}_q', \mathbf{r}_q')$ are symmetric. Applying this argument inductively, we get a tuple of the claimed form. By Lemma 23, (3.2) follows. By (3.1), (3.2) and Lemma 32, $(\mathbf{l}_q^*, \mathbf{q}, \mathbf{r}_q^*)$ and $(\mathbf{l}_q^*, \mathbf{r}_q^*)$ are symmetric.

Finally, we prove the uniqueness of $\mathbf{t}$, $\mathbf{l}_q^*$, $\mathbf{r}_q^*$. Suppose that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ is equivalent to another tuple $(rev(\mathbf{t}''), \mathbf{l}_q'', \mathbf{q}, \mathbf{r}_q'', \mathbf{t}'')$, where $\mathbf{l}_q''$ and $\mathbf{r}_q''$ are disjoint. By Lemma 23, $\mathbf{r}_q \sim (\mathbf{r}_q'', \mathbf{t}'') \sim (\mathbf{r}_q^*, \mathbf{t})$. Analogously, $\mathbf{l}_q \sim (rev(\mathbf{t}''), \mathbf{l}_q'') \sim (rev(\mathbf{t}), \mathbf{l}_q^*)$. Since $\mathbf{l}_q^*$ and $\mathbf{r}_q^*$ (resp. $\mathbf{l}_q''$ and $\mathbf{r}_q''$) are disjoint, it follows that the indices in $\mathbf{t}$ (resp. $\mathbf{t}''$) are precisely those indices, counting multiplicities, that occur in both $\mathbf{l}_q$ and $\mathbf{r}_q$. Thus, $\mathbf{t}''$ and $\mathbf{t}$ have the same indices. Because $(\mathbf{r}_q'', \mathbf{t}'') \sim (\mathbf{r}_q^*, \mathbf{t})$, by Proposition 21, $\mathbf{t}'' \sim \mathbf{t}$ and $\mathbf{r}_q'' \sim \mathbf{r}_q^*$. Similarly, it can be deduced that $\mathbf{l}_q'' \sim \mathbf{l}_q^*$. ∎

Example 43 *Let* $\mathbf{q} = (6, 3:5, 2, 0:1)$, $\mathbf{l}_q = (3:5, 1:2, 0:1)$ *and* $\mathbf{r}_q = (3:4, 2:3, 0:1)$. *It is easy to check that* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ *and* $(\mathbf{l}_q, \mathbf{r}_q)$ *are both symmetric index tuples. Note that* $\mathbf{l}_q$ *and* $\mathbf{r}_q$ *are not disjoint. We have*

$$\mathbf{l}_q \sim ((3), (4:5, 1:2, 0:1)) \quad \textit{and} \quad \mathbf{r}_q \sim ((3:4, 2, 0:1), (3)).$$

Then,

$$(4:5,1:2,0:1) \sim ((4),(5,1:2,0:1)) \quad \textit{and} \quad (3:4,2,0:1) \sim ((3,2,0:1),(4)).$$

Also,

$$(5,1:2,0:1) \sim ((1),(5,2,0:1)) \quad \textit{and} \quad (3,2,0:1) \sim ((3,2,0),(1)).$$

After two more steps, we conclude that

$$\mathbf{l}_q \sim ((3,4,1,2,0),(5,1)) \quad \textit{and} \quad \mathbf{r}_q \sim ((3),(0,2,1,4,3)).$$

Thus, (3.1) holds with $\mathbf{t} = (0,2,1,4,3)$, $\mathbf{l}_q^* = (5,1)$, *and* $\mathbf{r}_q^* = (3)$.

In the previous lemma we expressed the tuple $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ in the form $(rev(\mathbf{t}), \mathbf{l}_q^*, q,$
$\mathbf{r}_q^*, \mathbf{t})$ with $\mathbf{l}_q^*$ and $\mathbf{r}_q^*$ disjoint. Next we find an admissible tuple $\mathbf{w}$ such that $(\mathbf{l}_q^*, q, \mathbf{r}_q^*) \sim (rev(\mathbf{t}'), \mathbf{w}, \mathbf{r}_w, \mathbf{t}')$, where $\mathbf{r}_w$ is the symmetric complement of $\mathbf{w}$.

Lemma 44 *Let* $\mathbf{q}$ *be a permutation of* $\{0,1,\ldots,h\}$, $h \geq 0$, *and* $\mathbf{l}_q, \mathbf{r}_q$ *be disjoint tuples with indices from* $\{0,...,h-1\}$. *Suppose that* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ *is a symmetric tuple satisfying the SIP and* $(\mathbf{l}_q, \mathbf{r}_q)$ *is symmetric. Then, there exist an admissible tuple* $\mathbf{w}$ *relative to* $\{0,1,...,h\}$ *and an index tuple* $\mathbf{t}$ *with indices from* $\{0,\ldots,h-1\}$ *such that*

$$(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t}) \tag{3.3}$$

and

$$(\mathbf{l}_q, \mathbf{r}_q) \sim (rev(\mathbf{t}), \mathbf{r}_w, \mathbf{t}), \tag{3.4}$$

where $\mathbf{r}_w$ *is the symmetric complement of* $\mathbf{w}$.

Proof. In order to make the proof clearer, we assume $h \geq 2$. For $h < 2$ the result can be easily checked. The proof is by induction on the number of strings in $csf(\mathbf{q})$. Let $csf(\mathbf{q}) = (B_s, ..., B_1, B_0)$, where B_i, $i = 0, 1, .., s$, are the strings of $csf(\mathbf{q})$. Assume that $s = 0$, that is, $csf(\mathbf{q})$ has only one string. Then, $\mathbf{q} = (0:h)$, which is an admissible tuple. Note that, because of the SIP, $\mathbf{l}_q = \emptyset$. Let $\mathbf{r}'_q$ be the symmetric complement of $\mathbf{q}$. By Lemma 40, $(\mathbf{q}, \mathbf{r}'_q)$ satisfies the SIP, is symmetric, and $\mathbf{r}'_q$ is symmetric. We now show that $\mathbf{r}_q \sim \mathbf{r}'_q$, which implies the result. By Lemma 34, $(\mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{q}, \mathbf{r}'_q)$ satisfy the SS property. By Proposition 21, it is enough to show that for any $0 \leq i < h$, the subtuples of $\mathbf{r}_q$ and $\mathbf{r}'_q$ with indices from $\{i, i+1\}$ are the same. Note that in both tuples the first and last indices are equal to i. Because of the SIP, $h-1$ occurs exactly once in $\mathbf{r}_q$ and $\mathbf{r}'_q$. Then, $h-2$ occurs exactly twice. In general, $h-k$ occurs exactly k times in $\mathbf{r}_q$ and $\mathbf{r}'_q$. Thus, the claimed subtuples of $\mathbf{r}_q$ and $\mathbf{r}'_q$ with indices from $\{i, i+1\}$ coincide for each i, which implies, by Proposition 21, that $\mathbf{r}_q \sim \mathbf{r}'_q$.

Assume now that $s > 0$, that is, $csf(\mathbf{q})$ has more than one string. Note that, by Lemma 33, $\mathbf{l}_q$ and $\mathbf{r}_q$ commute. In the rest of the proof we use some notation introduced in Subsection 2.1.

Case 1: Suppose that $\mathbf{q} = (h :_\downarrow 0)$. Then, by Lemma 34, the subtuple of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ with indices from $\{h-1, h\}$ must be of the form $(h-1, h, h-1)$, since $(h, h-1)$ is a subtuple of $\mathbf{q}$. Thus, $h-1 \in \mathbf{l}_q$. Note that, because of the SIP, $\mathbf{l}_q$ has at most one index equal to $h-1$. Applying Lemma 34 to the subtuple of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ with indices from $\{h-2, h-1\}$, we deduce that the subtuple of $\mathbf{l}_q$ with indices from $\{h-2, h-1\}$ is $(h-2, h-1, h-2)$. By repeating this argument we conclude that $\mathbf{l}_q \sim (\mathbf{l}'_q, h-1 :_\downarrow 0)$, for some tuple $\mathbf{l}'_q \subset \{0, \ldots, h-2\}$. Since $\mathbf{r}_q$ and $\mathbf{l}_q$ are disjoint and have indices from $\{0, \ldots, h-1\}$, we deduce that $\mathbf{r}_q = \emptyset$. Because $\mathbf{l}_q = (\mathbf{l}_q, \mathbf{r}_q)$ is symmetric, it follows that

$$\mathbf{l}_q \sim (0 : h-2, \mathbf{l}''_q, h-1 :_\downarrow 0),$$

for some symmetric tuple $\mathbf{l}''_q \subset \{0, \ldots, h-3\}$. Note that, because of the SIP, $h-1, h-2 \notin \mathbf{l}''_q$. Therefore,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (0 : h-2, \mathbf{l}''_q, h-1 :_\downarrow 0,\ h :_\downarrow 0) \\ &\sim (0 : h, \left(\mathbf{l}''_q, h-2 :_\downarrow 0\right), h-1 :_\downarrow 0).\end{aligned}$$

Because $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ is symmetric, so is $(\mathbf{l}''_q, h-2 :_\downarrow 0)$. Thus, the tuple $(\mathbf{l}''_q, h-2 :_\downarrow 0)$ satisfies the conditions of the theorem. By the induction hypothesis, there exist an admissible tuple $\mathbf{w}^*$ relative to $\{0, 1, ..., h-2\}$ and an index tuple $\mathbf{t}^*$ with indices from $\{0, 1, ..., h-3\}$ such that $(\mathbf{l}''_q, h-2 :_\downarrow 0) \sim (rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*)$, where $\mathbf{r}^*_w$ is the symmetric complement of $\mathbf{w}^*$ and $\mathbf{l}''_q \sim (rev(\mathbf{t}^*), \mathbf{r}^*_w, \mathbf{t}^*)$. Then,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (0 : h, rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*, h-1 :_\downarrow 0) \\ &\sim (0 : h-2, rev(\mathbf{t}^*), (h-1 : h, \mathbf{w}^*), (\mathbf{r}^*_w, h-1), \mathbf{t}^*, h-2 :_\downarrow 0),\end{aligned}$$

and (3.3) holds with $\mathbf{t} = (\mathbf{t}^*, h-2 :_\downarrow 0)$, $\mathbf{r}_w = (h-1, \mathbf{r}^*_w)$ and $\mathbf{w} = (h-1 : h, \mathbf{w}^*)$. Condition (3.4) can be easily verified.

Case 2: Suppose that $B_s = (h)$ and $|B_i| > 1$ for some $i = 0, \ldots, s-1$. Let $j < s$ be the largest integer such that $|B_s| = \cdots = |B_{s-j}| = 1$. Then,

$$csf(\mathbf{q}) = (h :_\downarrow h-j,\ h-r : h-j-1, B_{s-j-2}, ..., B_0),$$

for some $r > j+1$. By Lemma 34, using an argument similar to that in Case 1, $\mathbf{l}_q \sim (\mathbf{l}'_q, h-1 :_\downarrow h-j-1)$, for some tuple $\mathbf{l}'_q \subset \{0,\ldots,h-2\}$. Note that, because of the SIP, $h-1 \notin \mathbf{l}'_q$. Since $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric and, by Lemma 33, $\mathbf{l}_q$ and $\mathbf{r}_q$ commute, we have that $\mathbf{l}_q$ is also symmetric, which implies

$$\mathbf{l}_q \sim (h-j-1 : h-2, \mathbf{l}''_q, h-1 :_\downarrow h-j-1),$$

for some symmetric tuple $\mathbf{l}''_q \subset \{0,\ldots,h-3\}$. Note that, by the SIP, $h-1 \notin \mathbf{l}''_q$. Also, for $j > 0$, again by the SIP, $h-2 \notin \mathbf{l}''_q$; when $j = 0$ the same conclusion follows from the symmetry of $\mathbf{l}_q$. Therefore,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (h-j-1 : h-2, \mathbf{l}''_q, h-1 :_\downarrow h-j-1, \mathbf{q}, \mathbf{r}_q)\\ &\sim (h-j-1 : h, \left(\mathbf{l}''_q, h-2 :_\downarrow h-j-1, B'_{s-j-1}, B_{s-j-2}, \ldots, B_0, \mathbf{r}_q\right), h-1 :_\downarrow h-j-1),\end{aligned}$$

where $B'_{s-j-1} := B_{s-j-1}[1]$. Observe that, since $\mathbf{l}_q$ and $\mathbf{r}_q$ commute, so do $(h-1 :_\downarrow h-j-1)$ and $\mathbf{r}_q$. As $\mathbf{l}_q$ and $\mathbf{r}_q$ are disjoint and $h-1 \in \mathbf{l}_q$, by Lemma 33, $h-2, h-1 \notin \mathbf{r}_q$. Thus, the tuple $\left(\mathbf{l}''_q, h-2 :_\downarrow h-j-1, B'_{s-j-1}, B_{s-j-2}, \ldots, B_0, \mathbf{r}_q\right)$ satisfies the conditions of the theorem. By the induction hypothesis, there exist an admissible tuple $\mathbf{w}^*$ relative to $\{0,1,\ldots,h-2\}$ and an index tuple $\mathbf{t}^*$ with indices from $\{0,1,\ldots,h-3\}$ such that

$$\left(\mathbf{l}''_q, h-2 :_\downarrow h-j-1, B'_{s-j-1}, B_{s-j-2}, \ldots, B_0, \mathbf{r}_q\right) \sim (rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*)$$

and $(\mathbf{l}''_q, \mathbf{r}_q) \sim (rev(\mathbf{t}^*), \mathbf{r}^*_w, \mathbf{t}^*)$, where $\mathbf{r}^*_w$ is the symmetric complement of $\mathbf{w}^*$. Then,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (h-j-1 : h, rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*, h-1 :_\downarrow h-j-1)\\ &\sim (h-j-1 : h-2, rev(\mathbf{t}^*), (h-1 : h, \mathbf{w}^*), (\mathbf{r}^*_w, h-1), \mathbf{t}^*, h-2 :_\downarrow h-j-1),\end{aligned}$$

and (3.3) holds with $\mathbf{t} = (\mathbf{t}^*, h-2 :_\downarrow h-j-1)$, $\mathbf{r}_w = (h-1, \mathbf{r}^*_w)$ and $\mathbf{w} = (h-1 : h, \mathbf{w}^*)$. Condition (3.4) can be easily verified.

Case 3: Suppose that $B_s = (h-r : h)$, for some $r \geq 1$. By Lemma 34, using an argument similar to that in Case 1, $\mathbf{r}_q \sim (h-r : h-1, \mathbf{r}'_q)$ for some tuple $\mathbf{r}'_q \subset \{0,\ldots,h-2\}$. Because $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric and $\mathbf{l}_q$ and $\mathbf{r}_q$ commute, the index tuple $\mathbf{r}_q$ is symmetric, which implies

$$\mathbf{r}_q \sim (h-r : h-1, \mathbf{r}''_q, h-2 :_\downarrow h-r),$$

for some symmetric tuple $\mathbf{r}''_q \subset \{0,\ldots,h-3\}$. Note that, because of the SIP, $h-1 \notin \mathbf{r}''_q$. For $r > 1$, again by the SIP, $h-2 \notin \mathbf{r}''_q$; for $r = 1$ the same conclusion follows from the symmetry of $\mathbf{r}_q$. Therefore,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (\mathbf{l}_q, h-r:h, B_{s-1}, h-r:h-2, B_{s-2}, ..., B_0, h-1, \mathbf{r}'_q) \\ &\sim (h-r:h, \mathbf{l}_q, (B_{s-1}, h-r:h-2, B_{s-2}, ..., B_0), \mathbf{r}''_q, h-1 :_\downarrow h-r).\end{aligned}$$

Observe that, since $\mathbf{l}_q$ and $\mathbf{r}_q$ commute, so do $(h-r:h-1)$ and $\mathbf{l}_q$. Also, since $h-1$ is not in $\mathbf{l}_q$, h commutes with $\mathbf{l}_q$. As $\mathbf{l}_q$ and $\mathbf{r}_q$ are disjoint and $h-1 \in \mathbf{r}_q$, by Lemma 33, $h-2, h-1 \notin \mathbf{l}_q$. Thus, by the induction hypothesis, there exist an admissible tuple $\mathbf{w}^*$ relative to $\{0, 1, ..., h-2\}$ and an index tuple $\mathbf{t}^*$ with indices from $\{0, 1, ..., h-3\}$ such that

$$(\mathbf{l}_q, (B_{s-1}, h-r:h-2, B_{s-2}, ..., B_0), \mathbf{r}''_q) \sim (rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*)$$

and $(\mathbf{l}_q, \mathbf{r}''_q) \sim (rev(\mathbf{t}^*), \mathbf{r}^*_w, \mathbf{t}^*)$, where $\mathbf{r}^*_w$ is the symmetric complement of $\mathbf{w}^*$. Then,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (h-r:h, rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*, h-1 :_\downarrow h-r) \\ &\sim (h-r:h-2, rev(\mathbf{t}^*), (h-1:h, \mathbf{w}^*), (\mathbf{r}^*_w, h-1), \mathbf{t}^*, h-2 :_\downarrow h-r),\end{aligned}$$

and (3.3) holds with $\mathbf{t} = (\mathbf{t}^*, h-2 :_\downarrow h-r)$, $\mathbf{r}_w = (h-1, \mathbf{r}^*_w)$ and $\mathbf{w} = (h-1:h, \mathbf{w}^*)$. Condition (3.4) can be easily verified. ■

Example 45 *Consider the tuples $\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q$ given in Example 43. We showed that*

$$(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim ((rev(0,2,1,4,3), (5,1), (6,3:5,2,0:1), 3, (0,2,1,4,3)).$$

We also have

$$((5,1), (6,3:5,2,0:1), 3) \sim ((5:6,3:5,1:2,0:1), 3) \sim ((5:6,3:4,1:2,0), (5,1,3)).$$

Thus,

$$(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim ((rev(0,2,1,4,3), (5:6,3:4,1:2,0), (5,1,3), (0,2,1,4,3)).$$

Note that $(5:6,3:4,1:2,0)$ is an admissible index tuple and $(5,3,1)$ is the corresponding symmetric complement.

The next theorem is the main result of this section and provides a full characterization of the symmetric tuples $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfying the SIP, with $(\mathbf{l}_q, \mathbf{r}_q)$ symmetric, in terms of admissible tuples.

Theorem 46 *Let $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, h\}$, $h \geq 0$, and $\mathbf{l}_q, \mathbf{r}_q$ be index tuples with indices from $\{0, 1, ..., h-1\}$ such that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. Then, $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ is a symmetric tuple, with $(\mathbf{l}_q, \mathbf{r}_q)$ symmetric, if and only if there exist an admissible tuple $\mathbf{w}$ relative to $\{0, 1, .., h\}$ and a tuple $\mathbf{t}$ with indices from $\{0, 1, ..., h-1\}$ such that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ and $(\mathbf{l}_q, \mathbf{r}_q) \sim (rev(\mathbf{t}), \mathbf{r}_w, \mathbf{t})$, where $\mathbf{r}_w$ is the symmetric complement of $\mathbf{w}$.*

Proof. Assume that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ is a symmetric tuple, with $(\mathbf{l}_q, \mathbf{r}_q)$ symmetric. Then, the claim follows from Lemmas 42 and 44.

The converse follows from the fact that, by Lemma 40, $(\mathbf{w}, \mathbf{r}_w)$ and $\mathbf{r}_w$ are symmetric. ∎

Taking into account the previous theorem, to obtain all possible symmetric tuples $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfying the SIP and such that $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric, it is enough to consider all admissible tuples $\mathbf{w}$ and all tuples $\mathbf{t}$ such that $(rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ satisfies the SIP, where $\mathbf{r}_w$ is the symmetric complement of $\mathbf{w}$. Next we characterize all tuples $\mathbf{t}$ with such property.

Definition 47 *Let $\mathbf{w}$ be an admissible tuple relative to $\{0, 1, \ldots, h\}$, $h \geq 0$, and $\mathbf{r}_w$ be the symmetric complement of $\mathbf{w}$. We say that a tuple $\mathbf{t}$ with indices from $\{0, \ldots, h-1\}$ is* $\mathbf{w}$-compatible *if, for any index i occurring in both $\mathbf{r}_w$ and $\mathbf{t}$, the subtuple of $\mathbf{t}$ with indices from $\{i, i+1\}$ starts with $i+1$.*

Lemma 48 *Let $\mathbf{w}$ be an admissible tuple relative to $\{0, \ldots, h\}$, $h \geq 0$. Let $\mathbf{r}_w$ be the symmetric complement of $\mathbf{w}$ and $\mathbf{t}$ be a tuple with indices from $\{0, \ldots, h-1\}$. Then, $(rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ satisfies the SIP if and only if*

i) $\mathbf{t}$ *satisfies the SIP*

ii) $\mathbf{t}$ *is* $\mathbf{w}$*-compatible.*

Proof. Assume that $(rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ satisfies the SIP. By Remark 26, condition i) holds. Condition ii) follows because, by definition of $\mathbf{r}_w$, for any index i in $\mathbf{r}_w$, the subtuple of $\mathbf{r}_w$ with indices from $\{i, i+1\}$ finishes with i.

Assume that $\mathbf{t}$ satisfies the SIP and is $\mathbf{w}$-compatible. Since $(\mathbf{w}, \mathbf{r}_w)$ and $\mathbf{t}$ satisfy the SIP, it is enough to check that between any two indices equal to i, one appearing in $(\mathbf{w}, \mathbf{r}_w)$ and the other in $\mathbf{t}$, there is an index $i+1$. But this follows from ii). Note that if $i < h$ is in $\mathbf{w}$ but not in $\mathbf{r}_w$ then $i+1$ is in $\mathbf{r}_w$. ∎

Note that if $(rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ satisfies the SIP, because $h-1$ is in $\mathbf{r}_w$ and h is neither in $\mathbf{t}$ nor in $\mathbf{r}_w$, then $h-1$ is not in $\mathbf{t}$.

The next example describes, up to equivalence, all tuples $\mathbf{t}$ such that $(rev(\mathbf{t}), \mathbf{w},$
$\mathbf{r}_w, \mathbf{t})$ satisfies the SIP, for a given admissible tuple $\mathbf{w}$.

Example 49 *Consider the admissible tuple $\mathbf{w} = (5 : 6, 3 : 4, 0 : 2)$ and its symmetric complement $\mathbf{r}_w = (5, 3, 0, 1, 0)$. We describe, up to equivalence, the tuples $\mathbf{t}$ with indices from $\{0, \ldots, 5\}$ such that $(rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ satisfies the SIP. Note that $5 \notin \mathbf{t}$.*

Suppose that $4 \in \mathbf{t}$. *Then, because* $5 \notin \mathbf{t}$ *and* $\mathbf{t}$ *satisfies the SIP,* 4 *occurs exactly once. Thus the subtuple of* $\mathbf{t}$ *with indices from* $\{4\}$ *is of the form*

$$(4).$$

Suppose that $3 \in \mathbf{t}$. *Then, because* $3 \in \mathbf{r}_w$, *by Lemma 48,* $4 \in \mathbf{t}$ *and occurs before the first occurrence of* 3. *Thus, the subtuple of* $\mathbf{t}$ *with indices from* $\{3,4\}$ *is of the form*

$$(4,3).$$

Suppose that $2 \in \mathbf{t}$. *If* $3 \in \mathbf{t}$, *by the previous case, the subtuple of* $\mathbf{t}$ *with indices from* $\{2,3,4\}$ *has one of the following forms:*

$$(2,4,3), \qquad (2,4,3,2), \qquad (4,3,2).$$

If $3 \notin \mathbf{t}$, *the subtuple with indices from* $\{2,3,4\}$ *has one of the following forms:*

$$(2), \qquad (2,4).$$

Suppose that $1 \in \mathbf{t}$. *Then, by Lemma 48,* $2 \in \mathbf{t}$ *occurs before the first occurrence of* 1. *Thus, the subtuple of* $\mathbf{t}$ *with indices from* $\{1,2,3,4\}$ *has one of the following forms:*

$$\begin{array}{llll} (2,1,4,3), & (2,1,4,3,2), & (2,1,4,3,2,1), & \\ (2,4,3,2,1), & (4,3,2,1), & (2,1), & (2,1,4). \end{array}$$

Finally, suppose that $0 \in \mathbf{t}$. *Then, by Lemma 48,* $1 \in \mathbf{t}$ *occurs before the first occurrence of* 0. *Thus, the subtuple of* $\mathbf{t}$ *with indices from* $\{0,1,2,3,4\}$ *has one of the following forms:*

$$\begin{array}{lll} (2,1,0,4,3), & (2,1,0,4,3,2), & (2,1,0,4,3,2,1), \\ (2,1,4,3,2,1,0), & (2,1,0,4,3,2,1,0), & (2,4,3,2,1,0), \\ (4,3,2,1,0), & (2,1,0), & (2,1,4,0). \end{array}$$

The twenty three displayed tuples are all possible tuples $\mathbf{t}$, *up to equivalence, such that* $(rev(t), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ *satisfies the SIP.*

Chapter 4

Fiedler pencils with repetitions

Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k as in (1.1). The family of Fiedler pencils with repetition (FPR) was defined in (24). In this paper, we describe the FPR that are symmetric when $P(\lambda)$ is.

We start by defining the matrices $M_i(P)$, depending on the coefficients of the matrix polynomial $P(\lambda)$, which appear as factors of the coefficients of a FPR. These matrices $M_i(P)$ are presented as block matrices partitioned into $k \times k$ blocks of size $n \times n$. Unless the context makes it ambiguous, we will denote these matrices by M_i instead of $M_i(P)$.

Recall that we have defined

$$M_0 := \left[\begin{array}{c|c} I_{(k-1)n} & 0 \\ \hline 0 & -A_0 \end{array}\right], \quad M_{-k} := \left[\begin{array}{c|c} A_k & 0 \\ \hline 0 & I_{(k-1)n} \end{array}\right],$$

and

$$M_i := \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & -A_i & I_n & 0 \\ 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right], \quad i = 1, \dots, k-1. \tag{4.1}$$

The matrices M_i in (**??**) are always invertible and their inverses are given by

$$M_{-i} := M_i^{-1} = \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 \\ 0 & I_n & A_i & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right].$$

The matrices M_0 and M_{-k} are invertible if and only if A_0 and A_k, respectively, are.

Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple with indices from $\{0, 1, ..., k-1, -1$
$, ..., -k\}$. We denote $M_{\mathbf{t}} := M_{i_1} M_{i_2} \cdots M_{i_r}$. If $\mathbf{t}$ is empty, then $M_{\mathbf{t}} = I_{kn}$. We also use the following notation: $revtr(M_{\mathbf{t}}) = M_{i_1}^T \cdots M_{i_r}^T$.

Remark 50 *It is easy to check that the commutativity relations*

$$M_i(P) M_j(P) = M_j(P) M_i(P), \quad \text{for any } P(\lambda), \tag{4.2}$$

hold if and only if $||i| - |j|| \neq 1$.

Next we show that, for two tuples $\mathbf{t}_1$ and $\mathbf{t}_2$ satisfying the SIP, $M_{\mathbf{t}_1}(P) = M_{\mathbf{t}_2}(P)$ if and only if $\mathbf{t}_1 \sim \mathbf{t}_2$. We start with a technical lemma.

Lemma 51 *Let $B = (a : b)$ be a string, with $0 \leq a \leq b < k$, and suppose that $b > a$ if $a = 0$. Let $T = I_{n(k-q-1)} \oplus T_0$ and $T' = I_{n(k-q-1)} \oplus T_0'$, with $q < b$, be two $nk \times nk$ block-partitioned matrices which differ at least in the block in position (i, j) for some $k - q + 1 \leq i \leq k$ and $k - q \leq j \leq k$. Then, the products $M_B T$ and $M_B T'$ have the forms $I_{n(k-q-2)} \oplus T_1$ and $I_{n(k-q-2)} \oplus T_1'$, for some matrices T_1 and T_1', and differ at least by the block in position (s, j) for some $k - q \leq s \leq k$.*

Proof. A calculation shows that

$$M_B = \left[\begin{array}{c|c|ccc|c}
I_{n(k-b-1)} & 0 & 0 & \cdots & & 0 \\
\hline
0 & -A_b & & & & 0 \\
\vdots & -A_{b-1} & & & & \vdots \\
\vdots & \vdots & & I & & \vdots \\
0 & -A_a & & & & 0 \\
\hline
0 & I_n & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & 0 & \cdots & 0 & I_{n(a-1)}
\end{array}\right], \tag{4.3}$$

if $a \neq 0$, and

$$M_B = \left[\begin{array}{c|c|cc}
I_{n(k-b-1)} & 0 & 0 & \cdots \\
\hline
0 & -A_b & & \\
 & -A_{b-1} & & \\
\vdots & \vdots & & I \\
0 & -A_1 & & \\
\hline
0 & -A_0 & 0 & 0
\end{array}\right], \tag{4.4}$$

if $a = 0$. Let i, j be as in the statement. Note that the matrix M_B has exactly one block I_n in the ith column, say in row s. In fact, we have either $s = i - 1$ or $s = i$. Moreover, in the sth row of M_B all the blocks are 0 except possibly the one in column $k - b$. Since the blocks in position $(k - b, j)$ in T and T' are zero, it follows that the products $M_B T$ and $M_B T'$ differ by at least the block in position (s, j). Note that $k - q \leq s \leq k$, proving the claim. ■

Lemma 52 *Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two index tuples with the same indices from $\{0, 1, \ldots, h\}$, $0 \leq h < k$. Assume that $\mathbf{t}_1$ and $\mathbf{t}_2$ satisfy the SIP. Then, $\mathbf{t}_1$ is equivalent to $\mathbf{t}_2$ if and only if $M_{\mathbf{t}_1}(P) = M_{\mathbf{t}_2}(P)$ for any matrix polynomial $P(\lambda)$ of the form (1.1).*

Proof. By Remark 50, the matrices M_i and M_j commute for any matrix polynomial $P(\lambda)$ if and only if the indices i and j commute. Thus, if $\mathbf{t}_1 \sim \mathbf{t}_2$, then $M_{\mathbf{t}_1} = M_{\mathbf{t}_2}$.

Assume now that $M_{\mathbf{t}_1}(P) = M_{\mathbf{t}_2}(P)$ and $\mathbf{t}_1$ and $\mathbf{t}_2$ are not equivalent. Let $csf(\mathbf{t}_1) = (B_{m_1}, ..., B_1, B_0)$ and $csf(\mathbf{t}_2) = (\tilde{B}_{m_2}, ..., \tilde{B}_1, \tilde{B}_0)$. Let r be the largest positive integer such that $B_{m_1-i+1} = \tilde{B}_{m_2-i+1}$, $i = 1, \ldots, r$. Then, $M_{B_{m_1},...,B_{m_1-r+1}} = M_{\tilde{B}_{m_2},...,\tilde{B}_{m_2-r+1}}$. Since $\mathbf{t}_1$ and $\mathbf{t}_2$ have the same indices, we deduce that $(B_{m_1-r}, ..., B_0)$ and $(\tilde{B}_{m_2-r}, ..., \tilde{B}_0)$ have the same indices as well. By the SIP, the largest index in a tuple occurs exactly once, and, by definition of column standard form, it appears in the first string (counting from left to right). Assume that $B_{m_1-r} = (a : b)$ and $\tilde{B}_{m_2-r} = (a' : b)$ with $a \neq a'$. Note that $M_{B_{m_1-r}}$ is of the form (4.3) if $a \neq 0$, and of the form (4.4) if $a = 0$. Since $b > i$ for all $i \in (B_{m_1-r-1}, ..., B_0)$, we have that $M_{B_{m_1-r-1},...,B_0}$ is of the form

$$\begin{bmatrix} I_{n(k-b)} & 0 \\ 0 & \star \end{bmatrix}.$$

Therefore, the matrix $M_1 := M_{B_{m_1-r},...,B_0}$ is of the form

$$\left[\begin{array}{c|cc|c} I_{n(k-b-1)} & 0 & 0 & 0 \\ \hline 0 & -A_b & \star & 0 \\ 0 & -A_{b-1} & \star & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -A_a & \star & 0 \\ 0 & I_n & \star & 0 \\ \hline 0 & 0 & 0 & \star \end{array}\right]$$

if $a \neq 0$, and of the form

$$\left[\begin{array}{c|cc} I_{n(k-b-1)} & 0 & 0 \\ \hline 0 & -A_b & \star \\ 0 & -A_{b-1} & \star \\ \vdots & \vdots & \vdots \\ 0 & -A_1 & \star \\ 0 & -A_0 & \star \end{array}\right],$$

if $a = 0$. A similar form can be obtained for $M_2 := M_{\tilde{B}_{m_2-r},\ldots,\tilde{B}_0}$. Since $a \neq a'$, we deduce that $M_{B_{m_1-r},\ldots,B_0} \neq M_{\tilde{B}_{m_2-r},\ldots,\tilde{B}_0}$. Clearly, if $M_{B_{m_1},\ldots,B_{m_1-r+1}}$ is nonsingular, which happens if 0 is not an index in $(B_{m_1},\ldots,B_{m_1-r+1})$ or A_0 is nonsingular, we have $M_{\mathbf{t}_1} \neq M_{\mathbf{t}_2}$. However, otherwise, this fact is not immediate. To prove it, note that there is a block in position (i,j), with $k-b+1 \leq i \leq k$ and $j = k-b$, by which M_1 and M_2 differ. By Lemma 51, $M_{B_{m_1-r+1}}M_1$ and $M_{\tilde{B}_{m_2-r+1}}M_2$ are of the form $I_{n(k-b-2)} \oplus T_0$ and $I_{n(k-b-2)} \oplus T_0'$, respectively, for some matrices T_0 and T_0', and differ by the block in position (s,j) for some $k-b \leq s \leq k$. Now apply Lemma 51 again, with $B = B_{m_1-r+2}$, $T = M_{B_{m_1-r+1}}M_1$ and $T' = M_{\tilde{B}_{m_2-r+1}}M_2$. After $r-2$ more steps, we conclude that $M_{\mathbf{t}_1}(P)$ and $M_{\mathbf{t}_2}(P)$ are distinct. ■

Remark 53 *We observe that the previous lemma holds if $\mathbf{t}_1$ and $\mathbf{t}_2$ are tuples satisfying the SIP and with the same indices from $\{-k,-k+1,\ldots,-1\}$.*

We recall the definition of the family of FPR.

Definition 54 *(FPR). Let $P(\lambda)$ be a matrix polynomial of degree k, as in (1.1). Let $h \in \{0,1,\ldots,k-1\}$. Let $\mathbf{q}$ and $\mathbf{z}$ be permutations of $\{0,1,\ldots,h\}$ and $\{-k,-k+1,\ldots,-h-1\}$, respectively. Let $\mathbf{l}_q$ and $\mathbf{r}_q$ be index tuples from $\{0,1,\ldots,h-1\}$ such that $(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)$ satisfies the SIP. Let $\mathbf{l}_z$ and $\mathbf{r}_z$ be index tuples from $\{-k,-k+1,\ldots,-h-2\}$ such that $(\mathbf{l}_z,\mathbf{z},\mathbf{r}_z)$ satisfies the SIP. Then, the pencil*

$$\lambda M_{\mathbf{l}_q,\mathbf{l}_z,\mathbf{z},\mathbf{r}_z,\mathbf{r}_q} - M_{\mathbf{l}_q,\mathbf{l}_z,\mathbf{q},\mathbf{r}_z,\mathbf{r}_q} \tag{4.5}$$

is called a Fiedler pencil with repetition (FPR) *associated with $P(\lambda)$.*

We observe that $M_{\mathbf{l}_q}$ and $M_{\mathbf{r}_q}$ commute with each factor in $M_{\mathbf{l}_z}M_{\mathbf{z}}M_{\mathbf{r}_z}$. Analogously, $M_{\mathbf{l}_z}$ and $M_{\mathbf{r}_z}$ commute with each factor in $M_{\mathbf{l}_q}M_{\mathbf{q}}M_{\mathbf{r}_q}$.

A FPR as in (4.5) can be expressed as $M_{\mathbf{l}_q,\mathbf{l}_z}(\lambda M_{\mathbf{z}} - M_{\mathbf{q}})M_{\mathbf{r}_q,\mathbf{r}_z}$. The pencil $\lambda M_{\mathbf{z}} - M_{\mathbf{q}}$ is a proper generalized Fiedler pencil (1; 2). It is known that, given a matrix polynomial $P(\lambda)$, any proper generalized Fiedler pencil is a strong linearization of $P(\lambda)$ (5). Therefore, we have the following result.

Lemma 55 *Let $P(\lambda)$ be a matrix polynomial (regular or singular). Let L be a FPR as in (4.5). Then, L is a strong linearization of $P(\lambda)$, unless one of the following conditions holds:*

i) 0 *is an index in* $\mathbf{l}_q$ *or* $\mathbf{r}_q$ *and* A_0 *is singular;*

ii) $-k$ *is an index in* $\mathbf{l}_z$ *or* $\mathbf{r}_z$ *and* A_{-k} *is singular.*

Thus, in order to obtain our linearizations we will require that none of the conditions i) and ii) in Lemma 55 hold and, in this case, we will say that the FPR L satisfies the *nonsingularity conditions*.

We finish this section by observing that in (24) the coefficients of the FPR are products of the matrices RM_iR, instead of M_i, where R is the $nk \times nk$ matrix

$$R := \begin{bmatrix} 0 & & I_n \\ & \iddots & \\ I_n & & 0 \end{bmatrix}. \tag{4.6}$$

Therefore, if the linearizations in Definition 54 are multiplied on the left and on the right by the matrix R, the linearizations constructed in (24) are obtained.

Chapter 5

Symmetric Linearizations

In Theorem 57 in this section we characterize all FPR that are symmetric when the matrix polynomial $P(\lambda)$ of degree k is. We observe that an analog of Theorem 57 holds in the Hermitian case. Namely, if $P(\lambda)$ is a Hermitian matrix polynomial of degree k of the form (1.1), then the pencil $P(\lambda)$ given in (5.1) is a Hermitian strong linearization of $P(\lambda)$, provided that $L(\lambda)$ satisfies the nonsingularity conditions. The proof of this claim is similar to the one of Theorem 57, noting that a result analog to Lemma 56 holds in the Hermitian case, that is, if $\mathbf{t}$ is a tuple as in the lemma, then $M_{\mathbf{t}}$ is Hermitian for any Hermitian $P(\lambda)$ of degree k if and only if $\mathbf{t}$ is symmetric.

Recall that $P(\lambda)$ is symmetric if and only if $A_i^T = A_i$, $i = 0, 1, ..., k$. Thus, when $P(\lambda)$ is symmetric, the matrices M_i and M_{-i} defined in Section 4 are symmetric for $i = 0, 1, ..., k$.

We start with a technical lemma. Recall the notation introduced in Section 4.

Lemma 56 *Let* $\mathbf{t}$ *be a tuple with indices from* $\{0, 1, ..., h\}$, $0 \leq h < k$. *Then,* $M_{\mathbf{t}}$ *is a symmetric matrix for any symmetric matrix polynomial* $P(\lambda)$ *of degree* k *if and only if* $\mathbf{t}$ *is symmetric.*

Proof. Assume that $\mathbf{t}$ is symmetric. Then, by Lemma 52, $M_{\mathbf{t}} = M_{rev(\mathbf{t})}$, which implies $M_{\mathbf{t}}^T = revtr(M_{rev(\mathbf{t})}) = revtr(M_{\mathbf{t}}) = M_{\mathbf{t}}$, where the last equality follows from the fact that $P(\lambda)$ is symmetric.

Assume now that $M_{\mathbf{t}}$ is symmetric. Then, $M_{\mathbf{t}} = M_{\mathbf{t}}^T = revtr(M_{rev(\mathbf{t})}) = M_{rev(\mathbf{t})}$. Then, by Lemma 52 again, the result follows. ■

Consider the FPR $L(\lambda)$ given in (4.5), associated with the matrix polynomial $P(\lambda)$, as in (1.1). Then, $L(\lambda)$ is symmetric if and only if

$$(M_{\mathbf{l}_q,\mathbf{l}_z,\mathbf{z},\mathbf{r}_z,\mathbf{r}_q})^T = M_{\mathbf{l}_q,\mathbf{l}_z,\mathbf{z},\mathbf{r}_z,\mathbf{r}_q}$$

and

$$(M_{\mathbf{l}_q,\mathbf{l}_z,\mathbf{q},\mathbf{r}_z,\mathbf{r}_q})^T = M_{\mathbf{l}_q,\mathbf{l}_z,\mathbf{q},\mathbf{r}_z,\mathbf{r}_q},$$

or, equivalently,

$$(M_{\mathbf{l}_q,\mathbf{r}_q})^T = M_{\mathbf{l}_q,\mathbf{r}_q}, \quad (M_{\mathbf{l}_z,\mathbf{z},\mathbf{r}_z})^T = M_{\mathbf{l}_z,\mathbf{z},\mathbf{r}_z},$$

and

$$(M_{\mathbf{l}_z,\mathbf{r}_z})^T = M_{\mathbf{l}_z,\mathbf{r}_z}, \quad (M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q})^T = M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}.$$

Taking into account Lemma 56, it follows that $L(\lambda)$ is symmetric for any $P(\lambda)$ if and only if $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$, $(\mathbf{l}_q, \mathbf{r}_q)$, $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$, and $(\mathbf{l}_z, \mathbf{r}_z)$ are symmetric. Thus, because of Theorem 46 and Lemma 48, the following result produces all symmetric FPR strong linearizations of a symmetric matrix polynomial $P(\lambda)$.

Theorem 57 *Let $P(\lambda)$ be a symmetric matrix polynomial of degree k of the form (1.1) and $0 \leq h < k$. Let $\mathbf{w}$ and $\mathbf{w}'$ be admissible tuples relative to $\{0, \ldots, h\}$ and $\{0, \ldots, k-h-1\}$, respectively. Let $\mathbf{r}_w, \mathbf{r}_{w'}$ be the symmetric complements of $\mathbf{w}$ and $\mathbf{w}'$, respectively. Let $\mathbf{t}_w \subset \{0, \ldots, h-1\}$ and $\mathbf{t}_{w'} \subset \{0, \ldots, k-h-2\}$ be index tuples satisfying the SIP and such that $\mathbf{t}_w$ is $\mathbf{w}$ -compatible and $\mathbf{t}_{w'}$ is $\mathbf{w}'$-compatible. Let $\mathbf{z} = -k + \mathbf{w}'$, $\mathbf{r}_z = -k + \mathbf{r}_{w'}$ and $\mathbf{t}_z = -k + \mathbf{t}_{w'}$. Then, the pencil*

$$L(\lambda) = \lambda M_{rev(\mathbf{t}_w),rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z,\mathbf{r}_w,\mathbf{t}_w} - M_{rev(\mathbf{t}_w),rev(\mathbf{t}_z),\mathbf{w},\mathbf{r}_z,\mathbf{t}_z,\mathbf{r}_w,\mathbf{t}_w} \tag{5.1}$$

is a symmetric FPR. Moreover, if $L(\lambda)$ satisfies the nonsingularity conditions, then $L(\lambda)$ is a strong linearization of $P(\lambda)$.

Proof. By Lemma 48 and Remark 26, the tuples $(rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w)$ and $(rev(\mathbf{t}_z), \mathbf{z}, \mathbf{r}_z, \mathbf{t}_z)$ satisfy the SIP. Thus, (5.1) is a FPR. We now show that the pencil $L(\lambda)$ is symmetric. Note that

$$L(\lambda) = \lambda M_{rev(\mathbf{t}_z,\mathbf{t}_w)} M_{\mathbf{z},\mathbf{r}_z,\mathbf{r}_w} M_{\mathbf{t}_z,\mathbf{t}_w} - M_{rev(\mathbf{t}_z,\mathbf{t}_w)} M_{\mathbf{w},\mathbf{r}_w,\mathbf{r}_z} M_{\mathbf{t}_z,\mathbf{t}_w}.$$

Since $P(\lambda)$ is symmetric, we have $M_i^T = M_i$, $i = 0, \ldots, k-1$ and $M_{-i}^T = M_{-i}$, $i = 1, ..., k$. Thus, $L(\lambda)$ is symmetric if

$$L'(\lambda) = \lambda M_{\mathbf{z},\mathbf{r}_w,\mathbf{r}_z} - M_{\mathbf{w},\mathbf{r}_w,\mathbf{r}_z} \tag{5.2}$$

is symmetric. Taking into account Lemma 56, $L'(\lambda)$ is symmetric if $(\mathbf{w}, \mathbf{r}_w)$, $(\mathbf{z}, \mathbf{r}_z)$, $\mathbf{r}_w$ and $\mathbf{r}_z$ are symmetric, which holds by Lemma 40 and Remark 41. The second claim follows from Lemma 55. ∎

Table 5.1 Example 59

$\mathbf{w}$	$\mathbf{r}_w$	$\mathbf{t}_w$
0	$\emptyset$	$\emptyset$
(0:1)	(0)	$\emptyset$
(0:2)	(0:1,0)	$\emptyset$
(1:2, 0)	(1)	$\emptyset$, (0)
(0:3)	(0:2, 0:1, 0)	$\emptyset$
(2:3,0:1)	(2, 0)	$\emptyset$, (1), (1,0)

Remark 58 *When k is even and both coefficients A_0 and A_k of $P(\lambda)$ are both singular, no pencil $L(\lambda)$ given in Theorem 57 satisfies the nonsingularity conditions, since h and $k-h-1$ cannot be both even and, therefore, either $\mathbf{w}$ or $\mathbf{w}'$ has odd index, which implies that either $-k$ is in $\mathbf{r}_z$ or 0 is in $\mathbf{r}_w$. Thus, in this case there are no symmetric FPR that are strong linearizations of $P(\lambda)$. If k is even and not both A_0 and A_k are singular, Theorem 57 produces symmetric strong linearizations. In fact, if A_0 is singular and A_k is nonsingular, by choosing h even, $\mathbf{w}$ of index 0 and $\mathbf{t}_w$ not containing 0, the pencil (5.1) satisfies the nonsingularity conditions. If A_0 is nonsingular and A_k is singular, by choosing h odd (so that $k-h-1$ is even), $\mathbf{w}'$ of index 0 and $\mathbf{t}_{w'}$ not containing 0, the pencil (5.1) satisfies the nonsingularity conditions. When k is odd Theorem 57 produces symmetric strong linearizations for any symmetric $P(\lambda)$ of degree k.*

We finish this section with an application of Theorem 57.

Example 59 *Let $P(\lambda)$ be a symmetric matrix polynomial of degree $k=4$. We construct all possible symmetric strong linearizations of $P(\lambda)$ in the family of FPR. We assume that A_0 (resp. A_{-k}) is invertible if 0 is an index in $(\mathbf{r}_w, \mathbf{t}_w)$ (resp. $-k$ is an index in $(\mathbf{r}_z, \mathbf{t}_z)$), so that each given pencil satisfies the nonsingularity conditions. The possible admissible tuples $\mathbf{w}$ and their corresponding symmetric complements are given in Table 5.1. We also give the possible $\mathbf{w}$-compatible tuples $\mathbf{t}_w$ in each case. In Table 5.2, we give the possible tuples $\mathbf{z}$, $\mathbf{r}_z$, and $\mathbf{t}_z$.*

Thus, the appropriate combination of the tuples in Tables 5.1 and 5.2 produces, in total, ten distinct symmetric FPR.

Next we give the explicit expression of these pencils. We first list the four linearizations in the basis of $\mathbb{DL}(P)$ given in (18).

Table 5.2 Example 59

$\mathbf{z}$	$\mathbf{r}_z$	$\mathbf{t}_z$
-4	$\emptyset$	$\emptyset$
(-4:-3)	(-4)	$\emptyset$
(-4:-2)	(-4:-3, -4)	$\emptyset$
(-3:-2, -4)	(-3)	$\emptyset$, (-4)
(-4:-1)	(-4:-2, -4:-3, -4)	$\emptyset$
(-2:-1, -4:-3)	(-2, -4)	$\emptyset$, (-3), (-3, -4)

- *Let* $\mathbf{w} = (0)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-4 : -1)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L(\lambda) = \lambda \begin{bmatrix} 0 & 0 & 0 & A_4 \\ 0 & 0 & A_4 & A_3 \\ 0 & A_4 & A_3 & A_2 \\ A_4 & A_3 & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & A_4 & 0 \\ 0 & A_4 & A_3 & 0 \\ A_4 & A_3 & A_2 & 0 \\ 0 & 0 & 0 & -A_0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (0 : 1)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-4 : -2)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L(\lambda) = \lambda \begin{bmatrix} 0 & 0 & A_4 & 0 \\ 0 & A_4 & A_3 & 0 \\ A_4 & A_3 & A_2 & 0 \\ 0 & 0 & 0 & -A_0 \end{bmatrix} - \begin{bmatrix} 0 & A_4 & 0 & 0 \\ A_4 & A_3 & 0 & 0 \\ 0 & 0 & -A_1 & -A_0 \\ 0 & 0 & -A_0 & 0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (0 : 2)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-4 : -3)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L(\lambda) = \lambda \begin{bmatrix} 0 & A_4 & 0 & 0 \\ A_4 & A_3 & 0 & 0 \\ 0 & 0 & -A_1 & -A_0 \\ 0 & 0 & -A_0 & 0 \end{bmatrix} - \begin{bmatrix} A_4 & 0 & 0 & 0 \\ 0 & -A_2 & -A_1 & -A_0 \\ 0 & -A_1 & -A_0 & 0 \\ 0 & -A_0 & 0 & 0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (0 : 3)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-4)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L(\lambda) = \lambda \begin{bmatrix} A_4 & 0 & 0 & 0 \\ 0 & -A_2 & -A_1 & -A_0 \\ 0 & -A_1 & -A_0 & 0 \\ 0 & -A_0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -A_3 & -A_2 & -A_1 & -A_0 \\ -A_2 & -A_1 & -A_0 & 0 \\ -A_1 & -A_0 & 0 & 0 \\ -A_0 & 0 & 0 & 0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (0)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-2:-1, -4:-3)$, $\mathbf{t}_z = (-3)$. *Then, we get*

$$L(\lambda) = \lambda \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & A_4 & A_3 \\ 0 & A_4 & A_3 & A_2 \\ I & A_3 & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & A_4 & A_3 & 0 \\ I & A_3 & A_2 & 0 \\ 0 & 0 & 0 & -A_0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (0:1)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-3:-2, -4)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L(\lambda) = \lambda \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & A_4 & A_3 & 0 \\ I & A_3 & A_2 & 0 \\ 0 & 0 & 0 & -A_0 \end{bmatrix} - \begin{bmatrix} 0 & I & 0 & 0 \\ I & A_3 & 0 & 0 \\ 0 & 0 & -A_1 & -A_0 \\ 0 & 0 & -A_0 & 0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (1:2, 0)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-4:-3)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L(\lambda) = \lambda \begin{bmatrix} 0 & A_4 & 0 & 0 \\ A_4 & A_3 & 0 & 0 \\ 0 & 0 & -A_1 & I \\ 0 & 0 & I & 0 \end{bmatrix} - \begin{bmatrix} A_4 & 0 & 0 & 0 \\ 0 & -A_2 & -A_1 & I \\ 0 & -A_1 & -A_0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (2:3, 0:1)$, $\mathbf{t}_w = (1)$, $\mathbf{z} = (-4)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L(\lambda) = \lambda \begin{bmatrix} A_4 & 0 & 0 & 0 \\ 0 & -A_2 & -A_1 & I \\ 0 & -A_1 & -A_0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} - \begin{bmatrix} -A_3 & -A_2 & -A_1 & I \\ -A_2 & -A_1 & -A_0 & 0 \\ -A_1 & -A_0 & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (0)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-2:-1, -4:-3)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L(\lambda) = \lambda \begin{bmatrix} 0 & 0 & A_4 & 0 \\ 0 & 0 & 0 & I \\ A_4 & 0 & A_3 & A_2 \\ 0 & I & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} A_4 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & A_2 & 0 \\ 0 & 0 & 0 & -A_0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (2:3, 0:1)$, $\mathbf{t}_w = (1, 0)$, $\mathbf{z} = (-4)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L(\lambda) = \lambda \begin{bmatrix} A_4 & 0 & 0 & 0 \\ 0 & -A_2 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & A_0 \end{bmatrix} - \begin{bmatrix} -A_3 & -A_2 & I & 0 \\ -A_2 & -A_1 & 0 & -A_0 \\ I & 0 & 0 & 0 \\ 0 & -A_0 & 0 & 0 \end{bmatrix}.$$

Chapter 6

Conclusions

In this work we have studied the Fiedler pencils with repetition which are symmetric whenever the matrix polynomial $P(\lambda)$ is. We have characterized all such pencils and have also given necessary and sufficient conditions for them to be strong linearizations of $P(\lambda)$. Additionally, when the matrix polynomial $P(\lambda)$ has degree k and the coefficients of the terms of degree 0 and k are nonsingular, our family is a nontrivial extension of the basis of the k-dimensional vector space $\mathbb{DL}(P)$ studied in (12; 18). Examples show that our family contains more than k linearly independent linearizations but it is still an open question the dimension of the vector space that this family generates for a general k. It is also an open question the characterization of all the pencils in this vector space which are strong symmetric linearizations of the matrix polynomial $P(\lambda)$ when $P(\lambda)$ is symmetric. Notice that for general $P(\lambda)$ this vector space consists of block-symmetric pencils.

Bibliography

[1] E. N. ANTONIOU AND S. VOLOGIANNIDIS, *A new family of companion forms of polynomial matrices*, Electron. J. Linear Algebra, 11 (2004), pp. 78–87.

[2] E. N. ANTONIOU AND S. VOLOGIANNIDIS, *Linearizations of polynomial matrices with symmetries and their applications*, Electron. J. Linear Algebra, 15 (2006), pp. 107–114.

[3] T. BETCKEA, D. KRESSNER *Perturbation, Extraction and Refinement of Invariant Pairs for Matrix Polynomials*, Linear Algebra Appl. 435 (2011), pp. 514-536.

[4] M.I. BUENO, AND S. FURTADO, *Palindromic linearizations of a matrix polynomial of odd degree obtained from Fiedler pencils with repetition*, Electron. J. Linear Algebra, 23 (2012), pp. 562-577.

[5] M.I. BUENO, F. DE TERAN, AND F. M. DOPICO, *Recovery of eigenvectors and minimal bases of matrix polynomials from generalized Fiedler linearizations*, SIAM. J. Matrix. Anal. Appl. 32(2011), pp. 463-483.

[6] F. DE TERÁN, F. M. DOPICO, AND D. S. MACKEY, *Structured linearizations for palindromic matrix polynomials of odd degree*, J.Comp. and Appl.Math, 236 (2011), pp. 1464–1480.

[7] M. FIEDLER *A note on companion matrices*, Linear Algebra Appl. 372 (2003), pp. 325-331.

[8] F. R. GANTMACHER, *The theory of matrices*, Chelsea, New York, 1959.

[9] I. GOHBERG, P. LANCASTER, AND L. RODMAN, *Matrix Polynomials*, Academic Press, New York, 1982.

[10] N. J. HIGHAM, R-C. LI, AND F. TISSEUR, *Backward error of polynomial eigenproblems solved by linearization*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 1218–1241.

[11] N. J. HIGHAM, D. S. MACKEY, AND F. TISSEUR, *The conditioning of linearizations of matrix polynomials*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1005–1028.

[12] N. J. HIGHAM, D. S. MACKEY, N. MACKEY, AND F. TISSEUR, *Symmetric linearizations for matrix polynomials*, SIAM Journal on Matrix Analysis and Applications, 29 (2006), pp. 143-159.

[13] A. HILLIGES, C. MEHL, AND V. MEHRMANN, *On the solution of palindromic eigenvalue problems*, in Proceedings of the 4th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS), Jyväskylä, Finland, 2004. CD-ROM.

[14] P. LANCASTER, *Symmetric transformations of the companion matrix*, NABLA Bull. Malay. Math. Soc., 8 (1961), pp. 146–148.

[15] P. LANCASTER *Lambda-Matrices and Vibrating Systems*. Pergamon Press, Oxford, UK, 1966. Reprinted by Dover, New York.

[16] P. LANCASTER AND U. PRELLS, *Isospectral families of high-order systems*, Z Angew. Math. Mech., 87(3) (2007), pp. 219–234.

[17] P. LANCASTER AND M. TISMENETSKY *The Theory of Matrices with Applications*. Academic Press, San Diego, CA, 1985.

[18] D. S. MACKEY, N. MACKEY, C. MEHL, AND V. MEHRMANN, *Vector spaces of linearizations for matrix polynomials*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 971–1004.

[19] D. S. MACKEY, N. MACKEY, C. MEHL, AND V. MEHRMANN, *Smith forms of palindromic matrix polynomials*, Electr. Journal of Linear Algebra, 22(2011), pp. 53–91.

[20] D. S. MACKEY, N. MACKEY, C. MEHL, AND V. MEHRMANN, *Skew-symmetric matrix polynomials and their Smith forms*, preprint.

[21] A. J. M. SPENCER, R. S. RIVLIN *The Theory of Matrix Polynomials and its Application to the Mechanics of Isotropic Continua* Rational Mechanics and Analyis, 2 (1958/9), pp. 309-336.

[22] F. TISSEUR, *Backward error and condition of polynomial eigenvalue problems*, Linear Algebra Appl., 309 (2000), pp. 339–361.

[23] F. TISSEUR AND K. MEERBERGEN, *The quadratic eigenvalue problem*, SIAM Review, 43 (2001), pp. 235–286.

[24] S. VOLOGIANNIDIS AND E. N. ANTONIOU, *A permuted factors approach for the linearization of polynomial matrices*, Math. Control Signals Syst., 22 (2011), pp. 317–342.