Presentation of the Motzkin Monoid

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Acknowledgements

The mathematical content described in this thesis is derived from a collaboration over the summer at the REU program here at UCSB, with Megan Ly (Loyola Marymount University), and Eliezer Posner (City College of New York).

Because of the REU setting, most of the work was done in amongst the three of us when we were together this summer at UCSB, in particular the material from section 4.1 onwards. I have appended a conclusion, written an introduction to the motzkin monoid (beginning of section 4), and have written almost all the introductory material before section 2.1.

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Abstract

In 2010, Halverson introduced the Motzkin algebra, a generalization of the Temperley-Lieb algebra, whose elements are diagrams that can be multiplied by stacking one on top of the other. Halverson gave a diagrammatic algorithm for decomposing any Motzkin diagram into diagrams of three subalgebras: the Right Planar Rook algebra, the Temperley-Lieb algebra, and the Left Planar Rook algebra. We first explore the Right and Left Planar Rook monoids, by finding presentations for these monoids by generators and relations, using a counting argument to prove that our relations suffice. We then turn to the newly-developed Motzkin monoid, where we describe Halverson's decomposition algorithm algebraically, find a presentation by generators and relations, and use a counting argument but with a much more sophisticated algorithm.

1 Introduction

In this paper, the diagram monoids R_n, and one of its submonoids P_n are first introduced in order to get an understanding of the diagram monoids RP_n, LP_n,which are submonoids of P_n. Presentations of the latter two are provided in this paper, and references to papers in which presentations for the former two are provided. We then introduce the well known Temperley-Lieb Algebra, TL_n. With these diagram monoids, and the Temperley-Lieb algebra, we can give a presentation of the Motzkin Monoid, M_n. In particular, we show any diagram in M_n can be decomposed into a product of the form RTL, where $r \in RP_n$, $t \in TL_n$, $l \in LP_n$ (Note there is a particular form called "standard" form mainly involving the placement of empty vertices). This is particularly important because of the inductive nature of our proof.

We start with a word in RTL form, append a generator, x_i to the end of the word, then try to get it back into RTL form. We apply the relations of M_n to first get $RTLx_i$ into P_1TP_2 (which we call PTP form, where $P_1, P_2 \in P_n$, $T \in T_n$. From this PTP form, we then get to Minimal RTL form, which is done by taking a diagram in PTP form, and moving all the empty vertices to the right. This is done using five lemmas (hop, burrow, slide, wallslide, and fuse), and also putting an order on PTP. From here, we concern ourselves with the case that the edges in T in minimal RTL form have endpoints that are empty vertices of R or L. We show that we can put this form word into standard RTL form (where standard RTL form is as defined in section 4.3.1).

The preceding inductive proof tells us that the number of distinct words is equal to the number of standard words, and thus the diagrams in M_n, giving us a presentation of of M_n.

1.1 Motivation

Diagram algebras and monoids show up in a number of mathematical fields such as statistical mechanics (within which the Temperley-Lieb algebra was introduced), knot theory, and algebraic groups. The planar rook algebra in particular was used by a fellow group of ours at UCSB during the REU, resulting in the paper *The Alexander and Jones Polynomials through representations of rook algebras* [8].

One of the important results of giving a presentation of a monoid, as is done in our paper, lies in representations (homomorphisms from one monoid, such as the motzkin monoid, into other monoids), and also give an understanding the monoid itself (it is much easier to get an understanding of a monoid in terms of generators and relations). We can construct these homomorphisms by taking the image of generator elements of one monoid, and checking to see if the relations are preserved in the image. For example, in section 2.4 we derive the presentation of the left planar rook monoid, LP_n from the generators of the right planar rook monoid, RP_n, and using an antiisomorphism $*$, we show the relations in RP_n are preserved in the image of $*$ (LP_n).

In order for a homomorphism from one monoid to another to be well-defined ($a = b$ implies under homomorphism ϕ, $\phi(a) = \phi(b)$), it is necessary that there be a presentation for the monoid. Clearly, for this to be the case you need a proof for the presentation, showing that given a set of sufficient generators, and relations among those generators, that the relations suffice to characterize the monoid [9].

To obtain some of the intuition behind the topic here are a few definitions:

Definition 1. An $m - tangle$ for $m \geq 0$ is a certain kind of diagram in the plane, as follows:
We label the unit disc D with $2m$ labeled points on its boundary containing a finite number of non-intersecting discs in its interior each with some number of labeled points on their boundaries. The points are then connected by smooth disjoint curves. These curves are between the interior discs and D (there may also exist a finite number of curves not connecting any two discs together). They must connect even-numbered boundary points to even numbered points and odd to odd. Shading of the connected components of the tangle may occur to create regions (as is done for the annular Temperley-Lieb algebra) in some ordering such that regions whose closures meet have different shadings, with each region being shaded (black) or unshaded. Note that the first unshaded region gets denoted with a $*$.

A more in depth definition is included in [11].

Definition 2. A *planar algebra* P is a family P_n of vector spaces together with a multilinear map corresponding to each tangle $T \in P$ from vector spaces (one for each internal disc of T) to the vector space of the boundary of T.

In addition, there are some axioms capturing the diagrammatic action of tangles. A complete definition is provided in [13].

In [11], Vaughan Jones shows how to associate a general planar algebra with a biparitite graph. This paper can help give some intuition to the rook monoids discussed in this paper.

Definition 3. An **algebraic structure** is a set S with one or more binary operations, $*_1, \ldots, *_n$, defined on all the elements of $S \times S$ and is denoted by $(S, *_1, \ldots, *_n)$.

Definition 4. A **monoid**, M, is a set with a binary operator $*$, satisfying:
1. $\forall x, y, z \in M$, $x * (y * z) = (x * y) * z$
2. $\exists\ e \in M$, such that $e * x = x * e = x,\ \forall x \in M$

We thus get that a monoid is a group without inverses.

Definition 5. A **submonoid** of a monoid, M, is a subset N of M such that:

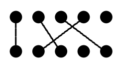
1. The identity, e, is contained in N.
2. If $x, y \in N$, then $xy \in N$.

Thus N is itself a monoid, and is closed under the operation of concatenation.

Definition 6. An **algebra** is a vector space V over a field F with a multiplication that is distributive, and:

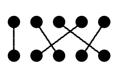
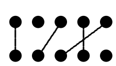
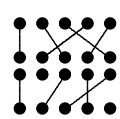
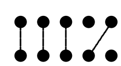
$$\forall f \in F, \text{ and } x, y \in V, f(xy) = (fx)y = x(fy)$$

We define the *Rook Monoid*, denoted R_n, as the set of one-to-one functions from a subset of $\{1, 2, ..., n\}$ to a subset of $\{1, 2, ..., n\}$. These functions can be written as diagrams: a graph on two rows of n vertices labeled 1 to n from left to right. On the graph we connect vertices on the top line to vertices on the bottom line. Take for example the following diagram in R_5:

$d =$ 

The Rook Monoid is a *monoid* under the operation of function composition, where the functions go from the top row of vertices to the bottom. To perform function composition with two diagrams d_1, d_2, we place d_1 above d_2 and identify the vertices in the bottom row of d_1 with the corresponding vertices in the top row of d_2. In terms of functions, if we consider d_1 as the function g, and d_2 as the function f, then the multiplication $d_1 d_2$ is thought of as the composition fg, where if x is in the range of g, and in the domain of f, we get that the pre-image of x, say y, in g is in the domain of fg, and the image of x, say z, in f is in the range of fg. In particular, y is in the domain of fg if and only if there exists x in the domain of f such that $g(y) = x$. This y then corresponds to the value z in the range of fg such that $f^{-1}(z) = x$. Note that in the diagram form we then draw an edge from y on top to z on bottom.

For example,

if $d_1 =$  and $d_2 =$  then $d_1 d_2 =$  $=$ 

Another way to think about the Rook Monoid, R_n is as a set of $n \times n$ matrices who have entries in the set $\{0, 1\}$, with the property that there is at most one 1 in each row and each column. Take for example, R_2, which consists of the matrices:

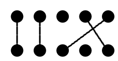
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{and} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note that the elements in R_n are in a one-to-one correspondence with the possible ways in which we can place non-attacking rooks on a $n \times n$ chess board. We define the *rank* of a diagram as its number of edges, or the number of 1s in the corresponding matrix.

If we desire a rook matrix of rank m, we choose m columns, and m rows in a total of $\binom{n}{k}^2$ ways. We then can choose to place the 1s in k of the chosen rows/columns in a total of $k!$ ways (such that no row or column has two or more nonzero entries). Summing over all the possible orders up to and including n gives rise to the order of R_n:

$$|R_n| = \sum_{k=0}^{n} \binom{n}{k}^2 k!$$

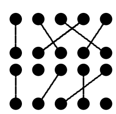
The relationship between diagrams and matrices is given in a very natural way. We connect the vertex in the i^{th} position in the top row of a diagram to the j^{th} position in the bottom row if and only if the corresponding matrix has a 1 in the (i,j)-position. Take for example the following matrix-diagram correspondence in R_5:



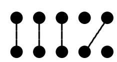
$$\leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Note that matrix multiplication is equivalent to diagram multiplication by stacking:

$d_1 =$
$d_2 =$



$=$



$= d_1 d_2$

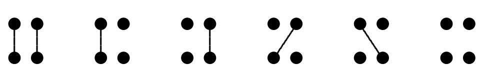
is equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

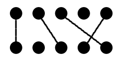
A presentation of the rook monoid is provided on page 339 in [6]

The *Planar Rook Monoid*, P_n, is the set of order preserving one-to-one functions from a subset of $\{1, 2, \ldots, n\}$ to a subset of $\{1, 2, \ldots, n\}$. The order preserving functions correspond to those diagrams that can be drawn with edges that do not cross. For example, the set P_2 consists of diagrams:

$P_2 =$



For a diagram d, we define $\tau(d)$ and $\beta(d)$ to be the sets containing the indices of the vertices of d which are incident to an edge on top and on bottom respectively. For example,

if $d =$  , then $\tau(d) = \{1,2,3,5\}$ and $\beta(d) = \{1',3',4',5'\}$

where we label the top vertices from 1 to n and the bottom vertices from $1'$ to n'. For a planar rook diagram d, there is only one way to connect the vertices by edges, thus these sets $\tau(d)$ and $\beta(d)$ uniquely determine d.

Notice that the product of two planar rook diagrams is planar (seen easily through diagram multiplication), thus P_n is a submonoid of R_n. To obtain the order of P_n, as in R_n, we choose k columns, and k rows in $\binom{n}{k}^2$, however as stated above this choosing leads to a unique diagram, thus the total number of planar rook diagrams is:

$$|P_n| = \sum_{k=0}^{n} \binom{n}{k}^2$$

1.2 Generators and Relations of the Planar Rook Monoid

Note: The following definitions are from *Contemporary Abstract Algebra.* [5]

Definition 7. For any set $S = \{a, b, c, \ldots\}$ of distinct symbols, we create a new set $S^{-1} = \{a^{-1}, b^{-1}, c^{-1}, \ldots\}$ by replacing each $x \in S$ by x^{-1}. Define the set $W(S)$ to be the collection of all formal finite strings of the form $x_1 x_2 \cdots x_k$, where each $x_i \in S \cup S^{-1}$. The elements of $W(S)$ are **words** coming from the set $W(S)$. Note that the *empty word* is also in $W(S)$, where the empty word is denoted by the identity, e.

We define $W(S)$ to be closed under the binary operation of concatenation, such that if $x_1 \cdots x_k$, $y_1 \cdots y_l \in W(S)$, then $x_1 \cdots x_k y_1 \cdots y_l \in W(S)$. Note this operation is associative, and the empty word is the identity. A string such as xx^{-1} is not the identity, because the elements of $W(S)$ are formal symbols with no implied meaning.

Definition 8. We say that $x \in W(S)$ is related to $y \in W(S)$ if y can be obtained from x by a finite sequences of insertions or deletions of words of the form aa^{-1}, or $a^{-1}a$, where $a \in S$.

Definition 9. A **monoid congruence** of M, $\sim$, is a subset $\sim \subseteq M \times M$ that is an equivalence relation, and $x \sim y$, $a \sim b \implies xa \sim yb \; \forall x, y, a, b \in M$.

Definition 10. We say that a monoid, M, is generated by a subset $A = \{a_1, \ldots a_k\}$ if and only if A is a submonoid of M, and any element of M can be written as the combination of finitely many elements of $W(A)$, under the operation of concatenation. If A is finite, then M is said to be finitely generated, and the elements of A are called **generators** of M.

Definition 11. Prior to defining a presentation, we must define a free monoid: Given a set of distinct symbols S, for any word u in $W(S)$, let $\overline{u}$ be all words in $W(S)$ equivalent to u (the equivalence class containing u). Then the set of all equivalence classes of elements of $W(S)$ is a monoid under the operation $\overline{u} \cdot \overline{v} = \overline{uv}$. This monoid is called a **free monoid** on S.

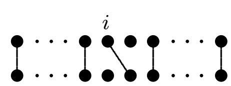
Definition 12. A **presentation** is defined in the following way:

Let M be a monoid generated by some subset $A = \{a_1, \ldots a_k\}$ and let F be the free monoid on A. Let $W = \{w_1, \ldots, w_t\}$ be a subset of F and let N be the smallest normal submonoid of F containing W,where a submonoid H is said to be normal if $Hx = xH \forall x \in M$. M is said to be given by *generators* $a_1, \ldots a_k$, and *relations* $w_1 = \cdots = w_t = e$ if there is an isomorphism from F/N onto M carrying $a_i N$ to a_i.

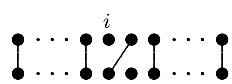
We then say that M has the **presentation**:

$$M =< a_1, \ldots, a_n | w_1 = \cdots = w_t = e >$$

Let l_i be the element of P_n such that $\tau(l_i) = [n] \setminus \{i+1\}$ and $\beta(l_i) = [n] \setminus \{i\} = \{1, 2, \ldots, i-1, i+i, \ldots, n\}$, as shown below:

$l_i =$ 

Let r_i be the element of P_n such that $\tau(r_i) = [n] \setminus \{i\}$ and $\beta(r_i) = [n] \setminus \{i+1\}$ as shown below:

$r_i =$ 

As proven in [2], every planar rook diagram can be written as a product of l_i's and r_i's.

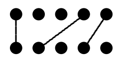
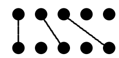
The following relations hold for all i such that the terms in the relation are defined:

1. $l_i^3 = l_i^2 = r_i^2 = r_i^3$
2. a) $l_i l_{i+1} l_i = l_i l_{i+1} = l_{i+1} l_i l_{i+1}$
   b) $r_i r_{i+1} r_i = r_{i+1} r_i = r_{i+1} r_i r_{i+1}$
3. a) $r_i l_i r_i = r_i$
   b) $l_i r_i l_i = l_i$
4. a) $l_{i+1} r_i l_i = l_{i+1} r_i$
   b) $r_{i-1} l_i r_i = r_{i-1} l_i$
   c) $r_i l_i r_{i+1} = l_i r_{i+1}$
   d) $l_i r_i l_{i-1} = r_i l_{i-1}$
5. $l_i r_i = l_{i+1} r_{i+1}$
6. If $|i - j| \geq 2$, then $r_i l_j = l_j r_i, r_i r_j = r_j r_i, l_i l_j = l_j l_i$

These relations can easily be verified by drawing the products they refer to. Furthermore, as proven in [2], these relations suffice to completely characterize P_n.

2 Right Planar Rook Monoid

We define the *Right Planar Rook Monoid* to be a submonoid of the Planar Rook Monoid. It has the property that the top vertex of each edge is directly above or above and to the right of the bottom vertex. Similarly, the *Left Planar Rook Monoid* is the submonoid where the top vertex of each edge is above or to the left of the bottom vertex. We denote these monoids RP_n and LP_n. For example,

$d_1 =$  and $d_2 =$ 

are in RP_5 and LP_5 respectively.

We will now derive some facts about RP_n.

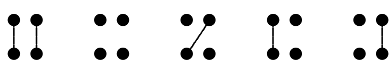
2.1 Cardinality

First, we prove a closed form for the size of RP_n. Through numerical experimentation we find that for the first few values of n, we get:

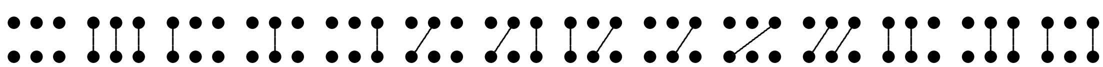
$$|RP_1| = 2, |RP_2| = 5, |RP_3| = 14, |RP_4| = 42, |RP_5| = 132.$$

$|RP_1|$ has diagrams:

$|RP_2|$ has diagrams:



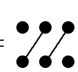
$|RP_3|$ has diagrams:



We notice that these cardinalities are Catalan numbers, and indeed we can prove that this pattern continues:

Theorem 1. $|RP_n| = C_{n+1} = \frac{\binom{2n+2}{n+1}}{n+2}$

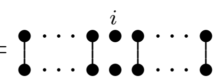
Proof. We begin by defining a new way to encode any diagram $d \in RP_n$. To do this we create ordered pairs of the form (i, j) for each edge in d connecting the ith vertex on top to the jth vertex on bottom. We then gather all such pairs into a set along with the pair $(n+1, n+1)$. We impose a well ordering on this set given by $(a, b) \leq (c, d) \iff a \leq c$. For example, the diagram given by:

$d =$ 

yields the poset $\{(2,1) \leq (3,2) \leq (4,4)\}$. Now we begin by exhibiting a bijection from RP_n to the set of all $2(n+1)$-sequences of ± 1s, with exactly $n+1$ of each, such that no partial sum is negative. The set of such sequences is known to have cardinality C_{n+1}. Define a function from RP_n to the set is as follows: given any diagram $d = \{(a_1, b_1) \leq \ldots \leq (a_k, b_k)\}$, start our sequence with a_1 1s followed by b_1 (-1)s. Inductively, append to this sequence $a_i - a_{i-1}$ 1s and $b_i - b_{i-1}$ (-1)s for $2 \leq i \leq k$. To see that the resulting sequence is indeed in our set, note that that diagrams in RP_n must consist of ordered pairs with their first component greater than or equal to their second. This, in combination with the fact that the first coordinate of every pair is appended first as 1s, makes it impossible to have a negative partial sum. To see that this function is a bijection, note that there is a natural inverse. Given any sequence, we create the first ordered pair by beginning from the left and counting off all of the 1s until the first (-1), as well as the number of (-1)s up to the next 1. These numbers will be the first and second coordinate of one's first ordered pair, (a_1, b_1). Inductively we count the ith set of 1s and the ith set of (-1)s. These numbers, a_i and b_i, respectively, would then be added to a_{i-1} and b_{i-1} to give an ordered pair (a'_i, b'_i). Therefore the cardinality of RP_n is equal to C_{n+1}. □

2.2 Generators

Let r_i be defined as in Subsection **??**. Let p_i be the element of RP_n such that $\tau(p_i) = \beta(p_i) = [n] \setminus \{i\}$ as shown:

$p_i =$ 

This yields the following theorem.

Theorem 2. *Every diagram of RP_n can be written as a product of r_i $(1 \leq i < n)$ and p_i $(1 \leq i \leq n)$.*

Proof. Suppose d is a diagram in RP_n with rank k. Let $\tau(d) = \{a_1, \ldots, a_k\}$ with $a_1 < \ldots < a_k$ and $\beta(d) = \{b_1, \ldots, b_k\}$ with $b_1 < \ldots < b_k$. Define R^a_a to be the identity diagram. For $1 \leq b < a \leq n$, define R^a_b to be

$$R^a_b = r_{a-1} \cdot r_{a-2} \ldots r_{b+1} \cdot r_b.$$

Then R^a_b is the diagram consisting of an edge from vertex b' to a and vertical edges connecting all vertices to the right of a or to the left of b'. Let,

$$d' = \prod_{i=1}^{k} R^{a_i}_{b_i} \cdot \prod_{i \notin \beta(d)} p_i\,.$$

It remains to prove that $d' = d$. Consider a node $b'_i \in \beta(d)$. All diagrams in $\prod_{i \notin \beta(d)} p_i$ have a vertical edge at b'_i. So do the diagrams $R^{a_j}_{b_j}$ for $j = k, k-1, \ldots, i+1$. Then $R^{a_i}_{b_i}$ connects nodes b'_i and a_i. Finally, $R^{a_j}_{b_j}$ has a vertical edge at a_i for $j = i-1, i-2 \ldots 1$. Thus d' connects nodes b'_i and a_i. If $j \notin \beta(d)$ then the product of p_i's ensures $j \notin \beta(d')$. Thus $d = d'$ is a word in our generators.

□

Note p_i is not considered a generator in P_n, since $p_i = r_i l_i$, for all $1 \leq i < n$ and that $p_i = l_{i-1} r_{i-1}$ for all $1 < i \leq n$.

2.3 Relations

Theorem 3. *The Right Planar Rook Monoid is generated by r_i and p_i subject to the following relations:*

1. $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$
2. $r_i^2 = r_i^3 = r_i p_i = p_i p_{i+1}$
3. $r_i = p_i r_i = r_i p_{i+1}$
4. $p_i r_{i+1} = r_{i+1} p_i$
5. $p_i^2 = p_i$
6. $p_i p_j = p_j p_i$
7. *If* $|i - j| \geq 2$ *then* $p_i r_j = r_j p_i$, *and* $r_i r_j = r_j r_i$

Proof. Theorem 2 shows that r_i and p_i generate all of RP_n, so every diagram can be written as a formal word. We must show that there are at most as many distinct equivalent classes of words as there are diagrams. To this end we define the standard word as follows:

A word W is said to be in standard form if there exist subsets of $[n]$, S and T with the following three properties:

• The sets S and T have equal cardinality, say k.

Let S_i and T_i be the ith elements S and T respectively. For example, if $S = \{1, 4, 7\}$, then $S_1 = 1, S_2 = 4, S_3 = 7$.

• For all $1 \leq i \leq k, T_i \leq S_i$.

Let R_b^a be defined as is Theorem 2.

• $W = RP$, where $R = \prod_{i=1}^{k} R_{T_i}^{S_i}$ and $P = \prod_{i \notin T} p_i$

Since every diagram corresponds to exactly one standard word, we must show that each formal word is equal to a standard word. Given a standard word $W = RP$ and a generator $x_i \in \{r_i, p_i\}$ we show that we can standardize the product $W x_i$. For our considerations, we will assume W is not the empty word, since if W is the empty word, even after appending either of the generators, W remains the empty word. First consider the product $W p_i$. We have two cases:

Case 1: $i \notin \beta(d)$. Then necessarily p_i appears in the product W. Since $p_i p_j = p_j p_i$, W can be written in the form: $W = RP_0 p_i$ where $P_0 = P \backslash p_i$. Then using the relation $p_j^2 = p_j$ we have that $W p_i = RP_0 p_i p_i = RP_0 p_i = W$.

Case 2: $i \in \beta(d)$. This means that $p_i \notin P$. So if $r_i \notin R$ then $W p_i = RP p_i = RP'$. If $r_i \in R$ then there exists j such that $r_i \in R_i^j$ for $i \leq j \leq k$. We commute the p_i next to the r_i and use the relation $r_s p_s = p_s p_{s+1}$ to remove the r's in R_i^j. Then we commute the p's past the r's. Thus,

$$\begin{aligned}
Wp_i &= RPp_i \\
&= R_{b_1}^{a_1} \dots R_i^j \dots R_{b_k}^{a_k} Pp_i \\
&= R_{b_1}^{a_1} \dots r_j \dots r_i p_i \dots R_{b_k}^{a_k} P \ (p_s p_t = p_t p_s \text{ and } p_s r_t = r_t p_s \text{ for } |t-s| \geq 2) \\
&= R_{b_1}^{a_1} \dots p_i \dots p_{j+1} \dots R_{b_k}^{a_k} P \ (r_s p_s = p_s p_{s+1}) \\
&= R_{b_1}^{a_1} \dots R_{b_k}^{a_k} Pp_i \dots p_{j+1} \ (p_s r_t = r_t p_s) \\
&= R'P'.
\end{aligned}$$

Now consider Wr_i. Let p_j be the rightmost letter of W. There are four cases to consider:

Case 1: $|i-j| \geq 2$. Since any of the generators commute when the indices are at least two apart, r_i can be moved to the left.

Case 2: $i = j-1$. Since $p_j \cdot r_{j-1} = p_{j-1} \cdot p_j$, we can eliminate r_i.

Case 3: $i = j+1$. Since $p_j \cdot r_{j+1} = r_{j+1} p_j$ we can move r_i to the left.

Case 4: $i = j$. Using the relation $p_j \cdot r_j = p_j \cdot p_{j+1}$, we can eliminate r_j.

These relations can then be applied repeatedly until the r's have been eliminated or commuted past P. If we eliminate the r's then we have that $Wr_i = RPr_i = RP'$. If the r's are commuted past P then we must show that Rr_lP can be standardized. Let r_j be the letter immediately to the left of r_l. We again have four cases depending on the difference between l and j:

Case 1: $|l-j| \geq 2$. Since any of the generators commute when the indices are at least two apart, r_l can be moved to the left.

Case 2: $l = j-1$. We have that $Rr_lP = R_{b_1}^{a_1} \cdots R_j^{a_m} r_{j-1} P = R_{b_1}^{a_1} \cdots R_{j-1}^{a_m} P$, which is already in standard form.

Case 3: $l = j+1$. This implies that $Rr_lP = R_{b_1}^{a_1} \cdots R_j^{a_m} r_{j+1} P$, which is already in standard form.

Case 4: $l = j$. Note that $r_l \cdot r_j = r_l^2 = p_l \cdot p_{l+1}$. Moving the $p_l \cdot p_{l+1}$ to the right, through r's, is equivalent to moving r's to the left through e's, which is proven above.

This process can be repeated as long as the word is not in standard form. So we have proven that Wx_i can be standardized. By induction, this implies that when a string of arbitrary length is appended to a word in standard form the product can be standardized. Thus any word can be put into standard form, implying that our relations hold, therefore are sufficient. □

2.4 Presentation of LP_n

For every diagram d in RP_n, let d^* is the diagram obtained by interchanging the vertices in the top and bottom rows of d while maintaining edge connections. Notice that

$$l_i^* = r_i,\ r_i^* = l_i,\ p_i^* = p_i$$

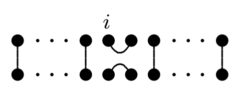
for all $1 \leq i < n$. Thus the function $*$ is an antiisomorphism and an involution, meaning that * is an isomorphism from RP_n to the opposite of RP_n (which is LP_n), and $*$ is its own inverse. Therefore LP_n is antiisomorphic to RP_n, and from every theorem about RP_n one can easily derive a corresponding theorem in LP_n. In particular, it can easily be shown that LP_n is generated by l_i and p_i and that it is completely characterized by relations analogous to those in Theorem 3

3 Temperley-Lieb Algebra

A *Temperley-Lieb diagram* has two rows of n vertices, connected by non-crossing edges. Unlike planar rook diagrams, Temperley-Lieb diagrams do not have any empty vertices and they may have *horizontal edges*. A horizontal edge is an edge connecting two distinct vertices in the same row. The collection of linear combinations of Temperley-Lieb diagrams forms the Temperley Lieb algebra, denoted $TL_n(x)$.

To multiply two Temperley-Lieb diagrams d_1 and d_2, we place d_1 on top of d_2, and identify the bottom vertices of d_1 with the top vertices of d_2. Additionally, horizontal edges between vertices in the bottom set of d_1 and top set of d_2 that form a closed loop produce a *bubble*. We eliminate the bubbles and multiply the product diagram by a factor of x for each bubble that is removed.

As discussed in [3], every Temperley-Lieb diagram can be written as the product of diagrams t_i, where

$t_i =$ 

for $1 \leq i < n$. The algebra $TL_n(x)$ is completely characterized by the following relations

$$\begin{aligned} t_i t_j &= t_j t_i \text{ if } |i-j| \geq 2 \\ t_i &= t_i t_j t_i \text{ if } |i-j| = 1 \\ t_i^2 &= x t_i \text{ for all } 1 \leq i < n. \end{aligned}$$

These relations are sufficient to show that any string of Temperley-Lieb generators is equivalent to a string of the form

$$x^m (t_{i_1} t_{i_1 - 1} \dots t_{j_1})(t_{i_2} t_{i_2 - 1} \dots t_{j_2}) \dots (t_{i_p} t_{i_p - 1} \dots t_{j_p}),$$

for some integers $m, p, i_1, j_1, \dots, i_p, j_p$ with the properties that

$$\begin{aligned} m, p &\geq 0 \\ 1 &\leq i_1 < i_2 < \cdots < i_p < n \\ 1 &\leq j_1 < j_2 < \cdots < j_p < n \\ j_1 &\leq i_1, j_2 \leq i_2, \dots, j_p \leq i_p. \end{aligned}$$

The above form is called the *standard form* of a Temperley-Lieb word.

We will need to use some non-standard forms of words in the Temperley-Lieb algebra.

Lemma 1. *For any Temperley-Lieb diagram, d, in which vertex i' is connected to j or j', we can express d as a word in one of the following five forms:*

1. *If i' is connected to j' with $i < j$. Then d can be written in the form $T(t_i t_{i+2} \cdots t_{j-1})(t_{i+1} t_{i+3} \cdots t_{j-2})T'$, where $T \in \langle t_1, \ldots t_{n-1} \rangle, T' \in \langle t_{i+1} \ldots t_{j-2} \rangle$. Note that if $j' = i' + 1$ then the word is of the form Tt_i.*

2. *If i' is connected to j' with $j < i$. Then d can be written in the form $T(t_j t_{j+2} \cdots t_{i-1})(t_{j+1} t_{j+3} \cdots t_{i-2})T'$, where $T \in \langle t_1, \ldots t_{n-1} \rangle, T' \in \langle t_{j+1} \ldots t_{i-2} \rangle$. Note that if $i' = j' + 1$ then the word is of the form Tt_j.*

3. *If i' is connected to j with $j < i$. Then d can be written in the form $T(t_{j+1} t_{j+3} \cdots t_{i-1})(t_j t_{j+2} \cdots t_{i-2})T'$, where $T \in \langle t_{j+1}, \cdots t_{n-1} \rangle, T' \in \langle t_1 \ldots t_{i-2} \rangle$.*

4. *If i' is connected to j with $j > i$. Then d can be written in the form $T(t_i t_{i+2} \cdots t_{j-2})(t_{i+1} t_{i+3} \cdots t_{j-1})T'$ where $T \in \langle t_1, \ldots t_{j-2} \rangle, T' \in \langle t_{i+1} \ldots t_{n-1} \rangle$.*

5. *If i' is connected to j on top with $j = i$. Then d can be written in the form TT', where $T \in \langle t_1, t_2, \ldots, t_{i-2} \rangle$, $T' \in \langle t_{i+1}, t_{i+2}, \ldots, t_{n-1} \rangle$.*

Proof. Given that these are the only five ways the vertex i' is connected to j or j', by drawing diagrams, one can see that these are indeed the ways in which Temperley-Lieb diagrams can be drawn. For example, in *1.*, simply take the diagram $t_i t_{i+2} \cdots t_{j-1}$ and perform multiplication by stacking it above the diagram $t_{i+1} t_{i+3} \cdots t_{j-2}$, along with any chosen diagram $T \in \langle t_1, \ldots t_{n-1} \rangle$ above this product, and $T' \in \langle t_{i+1} \ldots t_{j-2} \rangle$ below, and one can verify that indeed this gives the desired connection from i' to j' with $i < j$. □

In this paper we will consider multiplication of diagrams, but we will not consider linear combinations of diagrams. For simplicity, we will therefore take x to be 1 and refer exclusively to the monoid TL_n, although all results can be generalized to the algebra $TL_n(x)$, by multiplying by x where appropriate.

4 Motzkin Monoid

As seen in [4], for each $k = 0, 1, 2, \ldots$, the Motzkin numbers, $M^{(k)}$, are defined by the generating function:

$$M(t) = \sum_{k \geq 0} M^{(k)} t^n = \tfrac{1-t-\sqrt{1-2t-3t^2}}{2t^2}$$

Satisfying

$$M(t) = 1 + tM(t) + t^2M^2(t)$$

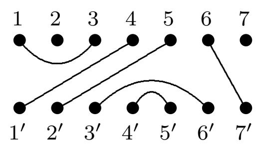
The Motzkin numbers appear in several places, including:

• The number of ways of drawing any number of nonintersecting chords among n points on a circle.

• The number of walks on $\{0, 1, \ldots\}$ with n steps from $\{-1, 0-1\}$ starting and ending at 0.

An extensive list can be found at: http://oeis.org/A001006

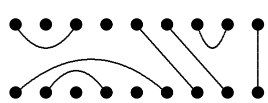
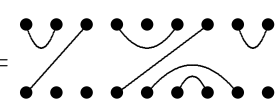
A Motzkin diagram is similar to a Temperley-Lieb diagram, however we may have *empty* vertices (vertices not incident to an edge). For example:

$d =$ 

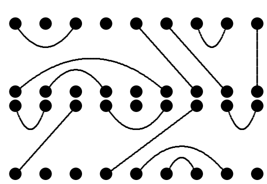
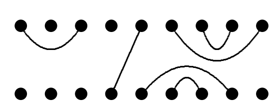
We define the *Motzkin Monoid*, M_n, as the set of Motzkin diagrams with n vertices. For example M_2 consists of the following diagrams:

$M_2 =$ 

Having horizontal edges complicates diagram multiplication. To multiply two Motzkin diagrams d_1 and d_2, we place d_1 on top of d_2, and identify the bottom vertices of d_1 with the top vertices of d_2. Additionally, horizontal edges between vertices in the bottom set of d_1 and top set of d_2 that form a closed loop produce a *bubble*. In diagram multiplication we eliminate bubbles, leaving vertices no longer incident to an edge. For example,

if $d_1 =$  and $d_2 =$ 

then

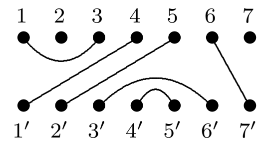
$d_1d_2 =$  $=$ 

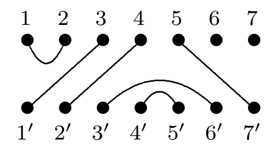
4.1 Decomposing the Motzkin Diagrams

In [1], it is proved that any diagram in the Motzkin monoid can be expressed in the form d=rtl where $r \in RP_n, t \in TL_n, l \in LP_n$. We now describe the algorithm to compute r, t, and l for any given d.

As in [2], we let $r = R^S$ be the diagram with top set, $\tau = S$ and bottom set, $\beta = \{1,\ldots,|S|\}$, and $l = L_T$ be the diagram with bottom set, $\beta = T$ and top set, $\tau = \{1,\ldots,|T|\}$.

To obtain t, first shift the isolated vertices of d to the right of the diagram, while preserving connections to the other vertices, to produce the diagram d'. For example, if d is the diagram below, then d' would be the diagram below it.

$d =$ 

$d' =$ 

Next we deal with the issue of filling in the empty vertices to complete the Temperley-Lieb diagram. Let τ_0 and β_0 denote the number of empty vertices on the top and bottom of d' respectively. For example, if d' is as above, then $\tau_0 = 2$ and $\beta_0 = 0$. Note that the difference between τ_0 and β_0 must even. We then have two cases for filling in the empty vertices:

Case 1: $|\tau(d)| \leq |\beta(d)|$. To turn d' into a Temperley-Lieb diagram, first add horizontal edges on top from j to $j+1$ for $j = |\tau(d)|+1, |\tau(d)|+3, \ldots, |\beta(d)|-1$. Now, add vertical edges from j to j' for $j = |\beta(d) + 1, \ldots, n$.

Case 2: $|\tau(d)| \geq |\beta(d)|$. To turn d' into a Temperley-Lieb diagram, first add horizontal edges on bottom from j' to $j' + 1$ for $j' = |\beta(d)| + 1, \ldots, |\tau(d)| - 1$. Now, add vertical edges from j to j' for $j = |\tau(d)| + 1, \ldots, n$

This gives a well defined way to decompose each diagram in the Motzkin Monoid. Thus $M_n = RP_n \; TL_n \; LP_n$.

4.2 Presentation of the Motzkin Monoid

Theorem 4. *The Motzkin Monoid is generated by t_i, r_i and r_i subject to the following relations.*

1. $r_i^3 = r_i^2 = l_i^2 = l_i^3$
2. a) $l_i l_{i+1} l_i = l_i l_{i+1} = l_{i+1} l_i l_{i+1}$
   b) $r_i r_{i+1} r_i = r_{i+1} r_i = r_{i+1} r_i r_{i+1}$
3. a) $l_i r_i l_i = l_i$
   b) $r_i l_i r_i = r_i$

4. a) $l_{i+1}r_il_i = l_{i+1}r_i$
   b) $r_{i-1}l_ir_i = r_{i-1}l_i$
   c) $r_il_ir_{i+1} = l_ir_{i+1}$
   d) $l_ir_il_{i-1} = r_il_{i-1}$

5. $l_ir_i = r_{i+1}l_{i+1}$

6. If $|i-j| \geq 2,$ then $r_il_j = r_jl_i,\ r_ir_j = r_jr_i,\ l_il_j = l_jl_i,\ t_ir_j = r_jt_i,\ t_il_j = l_jt_i,\ t_jt_i = t_jt_i$

7. $t_i^2 = t_i$

8. If $|i-j| = 1,$ then $t_it_jt_i = t_i$

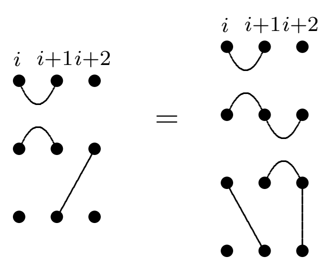
9. a) $t_il_i = t_ir_i$
   b) $l_it_i = r_it_i$

10. a) $t_ir_{i+1} = t_it_{i+1}l_i$
    b) $l_{i+1}t_i = r_it_{i+1}t_i$

11. a) $r_ir_{i+1}t_i = t_{i+1}r_ir_{i+1}$
    b) $t_il_{i+1}l_i = l_{i+1}l_it_{i+1}$

12. $t_il_it_i = t_i$

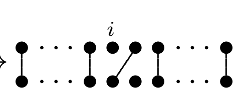
Each of these relations can be easily verified by drawing the diagrams in question. For example the picture below justifies the relation $t_ir_{i+1} = t_it_{i+1}l_i$.

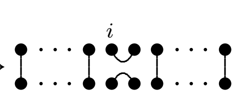


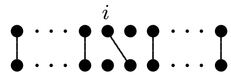
4.3 The Motzkin Monoid as Words

In this section we prove theorem 4. To start we will define a formal set of words which are subject exactly to the relations in theorem 4. Next we want to show that there is an isomorphism between this set and M_n.

We define M_n' to be the monoid generated by t_i, l_i, r_i subject to the same relations as above, and we let $\phi : M_n' \to M_n$ be the map given by

$r_i \mapsto$ 

$t_i \mapsto$ 

$l_i \mapsto$ 

Since the diagrams r_i, t_i, and r_i satisfy the relations on M_n, the operations are preserved; thus ϕ is a homomorphism. In order to prove that ϕ is also an isomorphism we want to show that M'_n has at most as many distinct elements as there are diagrams in M_n. We begin by defining a standard word in M'_n.

4.3.1 Standard Word

Given a word d in M'_n, let $\tau(d) = \{a_1, a_2, \ldots, a_k\}$ and $\beta(d) = \{b_1, b_2, \ldots, b_j\}$. We say d is in *standard form* if $d = RTL$ where

$$R = \prod_{i=1}^{k} R_i^{a_i} \cdot \prod_{i=k+1}^{n} p_i \text{ and } L = \prod_{i=j+1}^{n} p_i \cdot \prod_{i=0}^{j-1} L_{b_{j-i}}^{j-i}.$$

We then have three possibilities for T:

- If $k > j$ then, $T = T_1 t_{j+1} t_{j+3} \cdots t_{k-1}$ where $T_1 \in \langle t_1, t_2, \ldots, t_{k-1} \rangle$
- If $k < j$ then, $T = t_{k+1} t_{k+3} \cdots t_{j-1} T_1$ where $T_1 \in \langle t_1, t_2, \ldots, t_{j-1} \rangle$
- If $j = k$ then, $T = T_1$ where $T_1 \in \langle t_1, t_2, \ldots, t_{j-1} \rangle$ where T is in standard form for Temperley-Lieb words

This definition of a standard word corresponds to the diagram decomposition described by Halverson in [?]. By the definition of ϕ, M'_n maps to M_n. So if we show that we can standardize any formal word in M'_n then we will prove theorem 4. The following relations can be derived from the relations on M'_n, but will be useful:

- $r_i l_i = l_{i-1} r_{i-1} = p_i$
- $r_i = p_i r_i = r_i p_{i+1}$
- $l_i = p_{i+1} l_i = l_i p_i$
- $p_i l_i = p_{i+1} r_i = p_i p_{i+1}$
- $t_i r_i = t_i l_i = t_i p_i = t_i p_{i+1} = t_i p_i p_{i+1}$
- $r_i t_i = l_i t_i = p_i t_i = p_{i+1} t_i = p_i p_{i+1} t_i$

In order to prove that every element of M'_n is equivalent to a standard word, note that the identity is a standard word. Next, given a standard word W and a generator $x_i \in \{r_i, l_i, t_i\}$ we show that our relations are sufficient standardize the product Wx_i. In order to do this we proceed in three steps: First we get the word Wx_i into the form $P_1 T P_2$, known as PTP form, where $P_1, P_2 \in P_n$ and $T \in TL_n$. Next we manipulate PTP form in order to obtain minimal RTL form where $R \in RP_n, L \in LP_n, T \in TL_n$. After this, we manipulate the Temperley-Lieb diagram to obtain standard form.

4.3.2 PTP form

Lemma 2. *If RTL is a word in standard form and $x_i \in \{r_i, l_i, t_i\}$, then $RTLx_i$ is equal to a word in PTP form.*

Proof. Notice that appending an r_i or an l_i already yields a word in PTP form, so it suffices to show that $RTL \cdot t_i$ is equivalent to a word in PTP form. We will proceed using reverse induction on i.

For the base case we append t_{n-1}. There are several cases to consider depending on what the bottom set of L is:

Case 1: $(n-1)', n' \in \beta(L)$.
Because d is in standard form, there exist integers j and k, with $j \leq i$ and $j \leq k \leq n$ such that $L = L^{k,b_k} L^{k-1,b_{k-1}} \dots L^{j+1,i+1} L^{j,i} \dots L^{1,b_1}$. For simplicity, let $L_0 = L^{k,b_k} L^{k-1,b_{k-1}} \dots L^{j+2,b_{j+2}}$. Let $L_1 = L^{j-1,b_{j-1}} L^{j-2,b_{j-2}} \dots L^{1,b_1}$. Thus

$$L = L_0(l_{j+1} l_{j+2} \dots l_i)(l_j l_{j+1} \dots l_{i-1}) L_1.$$

By the relation $l_p l_q = l_q l_p$ for $|p-q| \geq 2$, we rearrange the terms to obtain $L = L_0(l_{j+1} l_j)(l_{j+2} l_{j+1}) \dots (l_i l_{i-1}) L_1$.

The indices of all the components of L_1 are strictly less than b_{j-1}, which is in turn strictly less than i, implying that the highest index of any of L_1's factors is less than or equal to $i-2$. Thus, t_i can commute past all these elements. Thus,

$$Lt_i = L_0(l_{j+1} l_j)(l_{j+2} l_{j+1}) \dots (l_i l_{i-1}) t_i L_1.$$

By repeatedly applying the relation $l_{p+1} l_p t_{p+1} = t_p l_{p+1} l_p$ we obtain

$$\begin{aligned} Lt_i &= L_0(l_{j+1} l_j)(l_{j+2} l_{j+1}) \dots l_{i-1} l_{i-2} t_{i-1} (l_i l_{i-1}) L_1 \\ &= L_0(l_{j+1} l_j)(l_{j+2} l_{j+1}) \dots (l_{i-2} l_{i-3}) t_{i-2} (l_{i-1} l_{i-2})(l_i l_{i-1}) L_1 \\ &\vdots \\ &= L_0 t_j (l_{j+1} l_j)(l_{j+2} l_{j+1}) \dots (l_i l_{i-1}) L_1 \end{aligned}$$

All the factors of L_1 have index greater than or equal to $j+2$, implying that t_j commutes with L_1. Therefore $Lt_i = t_j L$. Thus $dt_i = RTLt_i = RT'L$.

Case 2: $(n-1)', n' \notin \beta(L)$.

Since the $(n-1)'$ and n' vertices are empty, we have that $L = L_0 p_{n-1} p_n$ where $L_0 \in \langle l_1, \dots, l_{n-3}, p_1, \dots, p_{n-2} \rangle$. We have then,

$$RTLt_{n-1} = RTL_0 p_{n-1} p_n t_{n-1} = RT p_{n-1} p_n t_{n-1} L_0.$$

Now to get to PTP form we need to move $p_{n-1} p_n$. Note $n', (n-1)' \notin \beta(L)$, so $n, n-1 \notin \tau(L)$. Since RTL is in standard form, this tells us something about T. There are three subcases depending on $\beta(R)$.

Subcase 1: $(n-1)', n' \notin \beta(R)$.

Then $T \in \langle t_1, \ldots, t_{n-3} \rangle$. In this case every element of T commutes with $p_{n-1}p_nt_{n-1}$, and so

$$RTLt_{n-1} = RTp_{n-1}p_nt_{n-1}L_0 = Rp_{n-1}p_n(t_{n-1}T)L_0$$

Subcase 2: $(n-1)', n' \in \beta(R)$.

Then T can be written in the form T_0t_{n-1}, where $T_0 \in \langle t_1, \ldots, t_{n-2} \rangle$. So $T = T_0t_{n-1}$. Then

$$RTLt_{n-1} = RT_0t_{n-1}p_{n-1}p_nt_{n-1}L_0 = RT_0t_{n-1}L_0$$

Subcase 3: $(n-1)' \in \beta(R), n' \notin \beta(R)$.

Then T can be written as $T = T_0t_{n-2}$, where $T_0 \in \langle t_1, \ldots, t_{n-2} \rangle$. By our relations we get,

$$RTLt_{n-1} = RT_0t_{n-2}p_{n-1}p_nt_{n-1}L_0 = RT_0t_{n-2}l_{n-1}l_{n-2}L_0$$

In all three subcases, $RTLt_{n-1}$ is equal to a word in PTP form.

Case 3: $(n-1)' \in \beta(L), n' \notin \beta(L)$.
Since the nth vertex in L is empty, $L = Lp_n$. Using our relations, this implies that $RTLt_{n-1} = RTLp_nt_{n-1} = RTLp_{n-1}p_nt_{n-1}$. Now we are reduced to case 2.

Case 4: $(n-1)' \notin \beta(L), n' \in \beta(L)$. This case can be reduced to case 2 using analogous reasoning.

In conclusion, we have shown that if $RTL \in M_n$ is any word in standard form, then $RTLt_{n-1}$ is equivalent to a word in PTP form. We now move onto the inductive step to show that this holds for any t_i. Assume that for $j = i+1, i+2, ..., n-1$ $RTLt_j$ can be put into PTP form. We now show that this assumption implies that $RTLt_i$ can be put into PTP form. Consider the following three cases:

Case 1: $i', i'+1, i'+2 \notin \beta(L)$
Since $i, i+1, i+2 \notin \beta(L)$ we can write L as $L = Lp_ip_{i+1}p_{i+2}$. Thus,

$$\begin{aligned} RTLt_i &= RTLp_ip_{i+1}p_{i+2}t_i \\ &= RTLp_ip_{i+1}p_{i+2}t_{i+1}r_ir_{i+1} \\ &= (RTLp_ip_{i+1}p_{i+2}t_{i+1})r_ir_{i+1} \\ &= RTLt_{i+1}r_ir_{i+1}. \end{aligned}$$

The factor $RTLt_{i+1}$ right can be put into PTP form, by the inductive hypothesis. Thus we have $RTLt_i = P_1T'P_2r_ir_{i+1}$, which is in PTP form.

Case 2: $i', i'+1 \in \beta(L)$

We know that L can we written as $L = l_i l_{i-1} L_0$ where $r_0 \in \langle l_1 \ldots l_{i-2} \rangle$. So,

$$\begin{aligned} RTLt_i &= RTl_i l_{i-1} L_0 t_i \\ &= RTl_i l_{i-1} t_i L_0 \\ &= RTl_i r_i t_{i-1} l_i l_{i-1} L_0 \\ &= RTp_{i+1} t_{i-1} l_i l_{i-1} L_0 \\ &= RTt_{i-1} p_{i+1} l_i l_{i-1} L_0. \end{aligned}$$

Case 3: $i', i'+1 \notin \beta(L), i'+2 \in \beta(L)$

Since L is in standard form, and $i', i'+1 \notin \beta(L)$, we know that we cannot have a vertical line from $i+2$ to $i'+2$. Furthermore, $i', i'+1 \notin \beta(L)$ allows us to write L equivalently as $L = L_0 l_i l_{i+1}$ (because necessarily p_i, p_{i+1} are in our product L_0), where L_0 is defined by $\tau(L_0) = \tau(L)$, and$\beta(L_0) = (\beta(L) \setminus \{i+2\}) \cup \{i\}$. So we have,

$$\begin{aligned} RTLt_i &= RTL_0 l_i l_{i+1} t_i \\ &= (RTL_0 p_{i+1} p_{i+2} t_{i+1}) t_i. \end{aligned}$$

The factor in parenthesis can be put into $P_1 T P_2$ form, by the inductive hypothesis. Thus $RTLt_i = P_1 T' P_2 t_i$. Note that $\beta(P_2)$ contains $i, i+1$, and $i+2$. So $P_1 T' P_2 t_i$ can be put into standard form, by case 2.

Case 4: $i \in \beta(L), i+1 \notin \beta(L)$

Note that in this case $L = Lp_{i+1}$. Using our relations we have, $RTLt_i = RTLp_{i+1} t_i = RTLp_{i+1} p_i t_i$. Note that i and $i+1$ are not in $\beta(Lp_{i+1} p_i)$, implying that this word can be standardized by the previous case.

Case 5: $i \notin \beta(L), i+1 \in \beta(L)$

This case is analogous to case 4.

□

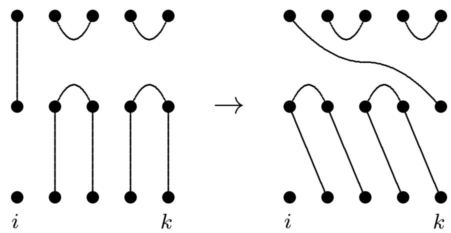
4.3.3 Minimal RTL form

Rewriting a word in PTP form as a word in standard form is analogous to taking a Motzkin diagram drawn as the product of a Temperley-Lieb diagram and two planar rook diagrams and redrawing it so that the empty vertices in the bottom of the top diagram and the empty vertices in the top of the bottom diagram are all on the far right. This motivates the following lemmas, each of which concerns words in M'_n but corresponds to shifting empty vertices rightward in diagrams in M_n.

Lemma 3 (Hop)**.** *Let i and k be natural numbers. If $i < k$ and $k - i$ is even then,*

$$(t_{k-1} t_{k-3} \ldots t_{i+1}) p_i = (t_{k-1} t_{k-3} \ldots t_{i+1})(t_{k-2} t_{k-4} \ldots t_i)(l_{k-1} l_{k-2} \ldots l_i).$$

Intuitively we are "hopping" a dead end in the ith vertex over a series of horizontal edges to the kth vertex as shown below:



Proof. We prove Lemma 3 by induction on k.

The case $k = i + 2$ is simply the relation $t_{i+1}p_i = t_{i+1}t_il_{i+1}l_i$.

Assume the result holds for $k - 2$:

$$(t_{k-3}t_{k-5}\dots t_{i+1})p_i = (t_{k-3}t_{k-5}\dots t_{i+1})(t_{k-4}t_{k-6}\dots t_i)(l_{k-3}l_{k-4}\dots l_i)$$

Then

$$\begin{aligned}
& (t_{k-1}t_{k-3}\dots t_{i+3}t_{i+1})p_i \\
= \; & t_{k-1}(t_{k-3}t_{k-5}\dots t_{i+1})(t_{k-4}t_{k-6}\dots t_i)(l_{k-3}l_{k-4}\dots l_i) \text{ (inductive hypothesis)} \\
= \; & t_{k-1}(t_{k-3}t_{k-5}\dots t_{i+1})(t_{k-4}t_{k-6}\dots t_i)p_{k-2}(l_{k-3}l_{k-4}\dots l_i) \; (l_i = p_{i+1}l_i) \\
= \; & (t_{k-3}t_{k-5}\dots t_{i+1})(t_{k-4}t_{k-6}\dots t_i)t_{k-1}p_{k-2}(l_{k-3}l_{k-4}\dots l_i) \text{ (commuting } t_{k-1}) \\
= \; & (t_{k-3}t_{k-5}\dots t_{i+1})(t_{k-4}t_{k-6}\dots t_i)t_{k-1}t_{k-2}l_{k-1}l_{k-2}(l_{k-3}l_{k-4}\dots l_i) \text{ (base case)} \\
= \; & (t_{k-1}t_{k-3}\dots t_{i+1})(t_{k-2}t_{k-4}\dots t_i)(l_{k-1}l_{k-2}\dots l_i) \text{ (commuting } t_{k-1} \text{ and } t_{k-2}).
\end{aligned}$$

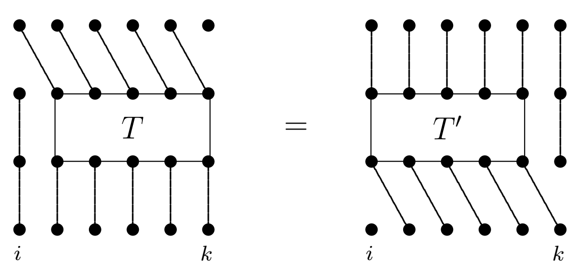
□

Lemma 4 (Slide)**.** *If* $T \in \langle t_{i+1}, t_{i+2}\dots, t_{k-1}\rangle$*, then*

$$(l_{k-1}l_{k-2}\dots l_i)T = T'(l_{k-1}l_{k-2}\dots l_i)$$

for some $T' \in \langle t_i, t_{i+1}\dots t_{k-2}\rangle$.

Diagrammatically we are "sliding" the product of l's under the product of t's, as shown below:



Proof. We will prove Lemma 4 using induction on the length of the word T.

Base case: For our base case we want to show that $l_{k-1}\dots l_i t_j = t_{j-1}l_{k-1}\dots l_i$ for some $t_j \in T$. Since $j \in [i+1, k-1]$ then there exists an l_{j-1} in our product of l's. We will commute t_j up to this point and then use our relations as follows:

$$\begin{aligned} l_{k-1}\dots l_i t_j &= l_k \dots l_{j-1}\dots l_i t_j \\ &= l_{k-1}\dots l_{j-2}l_{j-1}t_j l_{j-2}\dots l_i \ (l_i t_j = t_j l_i \text{ if } |i-j| \geq 2) \\ &= l_{k-1}\dots t_{j-1}l_{j-2}l_{j-1}l_{j-2}\dots l_i \ (t_i l_{i+1} l_i = l_{i+1} l_i t_{i+1}) \\ &= t_{j-1}l_k \dots l_i \ (l_i t_j = t_j l_i \text{ if } |i-j| \geq 2). \end{aligned}$$

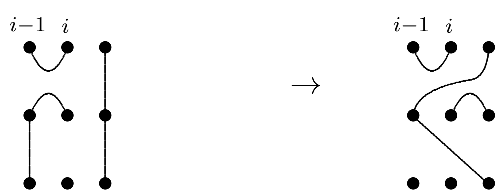
Induction: Now, assume $l_{k-1}\dots l_i T = T' l_{k-1}\dots l_i$ where T,T' are some products of t's as described above. If we append some $t_m \in T$, we have that:

$$\begin{aligned} l_{k-1}\dots l_i T t_m &= T' l_{k-1}\dots l_i t_m \text{ (inductive hypothesis)} \\ &= T' t_{m-1} l_{k-1}\dots l_{m-1}\dots l_i \text{ (base case)} \\ &= T'' l_{k-1}\dots l_i \end{aligned}$$

where $T'' \in < t_i, t_{i+1} \dots k-2 >$. So by induction we have that $(l_{k-1}l_{k-2}\dots l_i)T = T'(l_{k-1}l_{k-2}\dots l_i)$ as desired. □

Lemma 5 (Burrow)**.** *Let i be a natural number. Then, $t_{i-1}p_i = t_{i-1}t_i l_{i-1} l_i$.*

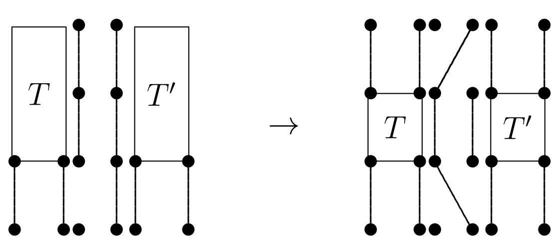
Intuitively, we will use this to allow a horizontal edge to "burrow under" a neighboring edge.



Proof. The proof of this lemma follows easily from the relations $p_i = r_i l_i$ and $t_i r_{i+1} = t_i t_{i+1} l_i$. Thus, $t_{i-1}p_i = t_{i-1}r_i l_i = t_{i-1}t_i l_{i-1} l_i$. □

Lemma 6 (Wallslide)**.** *Let i be a natural number. Then $TT'p_i = r_i TT' l_i$ where $t_j \in T$ has index less than i and $t_k \in T'$ has index greater than $i+1$.*

This is the case where we have a "punctured" vertical line (an edge from top vertex i to bottom vertex i), and want to move it over a non-punctured vertical line a wall as in the diagram:

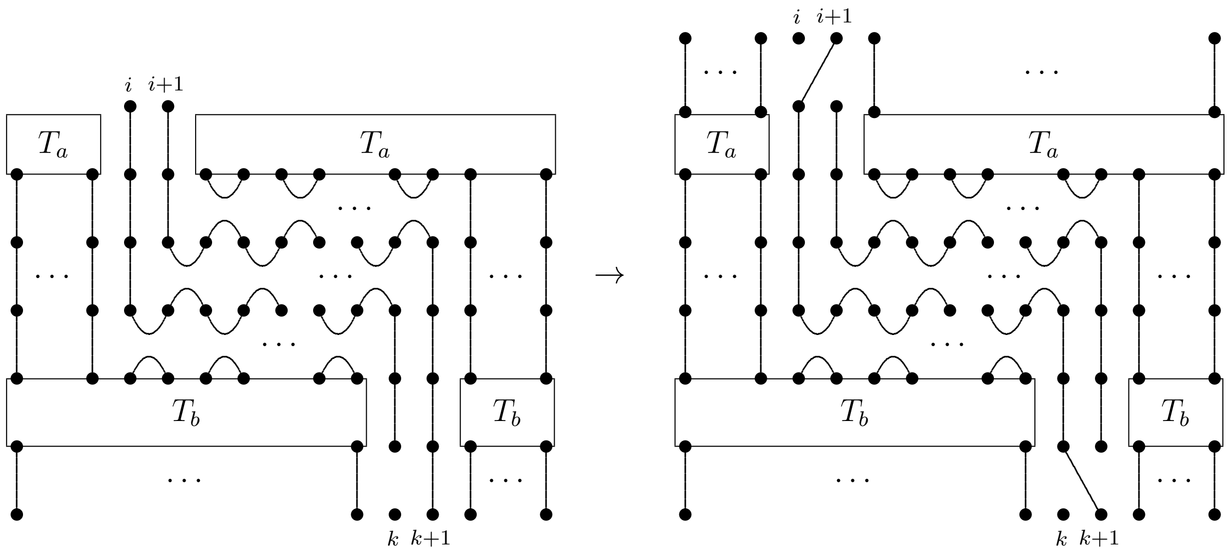


Proof. Since all of T's have index less than i and all of T''s factors have index greater than $i+1$, we can commute r_i to commute through T and T' to get

$$TT'e_i = TT'r_il_i = r_iTT'l_i.$$

□

If vertex i is connected to vertex j, then i being punctured is equivalent to j being punctured. This fact can be seen from the diagram and is stated formally and proven below.



Lemma 7 (Fuse)**.** *(i) If $j < i$, $i-j$ is even, and $x = (t_{j+1}t_{j+3}\dots t_{i-1})(t_jt_{j+2}\dots t_{i-2})$, then $xp_i = p_jx = p_ixp_j$.*
(ii) If $i < j$, $j-i$ is even, and $x = (t_it_{i+2}t_{i+4}\dots t_{j-2})(t_{i+1}t_{i+3}\dots t_{j-1})$, then $xp_i = p_jx = p_ixp_j$.
(iii) If $i < j$, $j-i$ is odd, and $x = (t_it_{i+2}\dots t_{j-1})(t_{i+1}t_{i+3}\dots t_{j-2})$, then $xp_j = xp_i = xp_jp_i$.
(iv) If $j < i$, $j-i$ is odd, and $x = (t_jt_{j+2}\dots t_{i-1})(t_{j+1}t_{j+3}\dots t_{i-2})$, then $xp_i = xp_j = xp_ip_j$.

Proof. (i) By the commutativity relations, we can rearrange x to be of the form

$$x = (t_{j+1}t_j)(t_{j+3}t_{j+2})(t_{j+5}t_{j+4})\dots(t_{i-3}t_{i-4})(t_{i-1}t_{i-2})$$

This implies that

$$\begin{aligned}
xp_i &= (t_{j+1}t_j)(t_{j+3}t_{j+2})\dots t_{i-5}t_{i-6}t_{i-3}t_{i-4}t_{i-1}t_{i-2}p_i \\
&= (t_{j+1}t_j)(t_{j+3}t_{j+2})\dots t_{i-3}t_{i-4}t_{i-1}p_it_{i-2} \ (t_jp_i = p_it_j \text{ for } |j-i|>1) \\
&= (t_{j+1}t_j)(t_{j+3}t_{j+2})\dots t_{i-3}t_{i-4}t_{i-1}p_{i-1}t_{i-2} \ (t_jp_{j+1} = t_jp_j) \\
&= (t_{j+1}t_j)(t_{j+3}t_{j+2})\dots t_{i-3}t_{i-4}t_{i-1}p_{i-2}t_{i-2} \ (e_{j+1}t_j = p_jt_j) \\
&= (t_{j+1}t_j)(t_{j+3}t_{j+2})\dots t_{i-3}t_{i-4}p_{i-2}t_{i-1}t_{i-2} \ (p_jt_{j+1} = t_{j+1}p_j)
\end{aligned}$$

This process can be repeated until xp_j has been transformed to $p_i x$. The equation $xp_j = p_i x p_j$ can be derived by first noting that $xp_j = xp_j^2$ and using the above result to move one of the factors of p_j to the far left.
(ii) The proof is analogous to that of the previous case.

(iii) $xp_j = t_i[t_{i+2}t_{i+4}\dots t_{j-1}t_{i+1}t_{i+3}\dots t_{j-2}p_j]$. Notice that the factor in brackets is of the form discussed in case (ii). Thus we obtain $xp_j = t_i[p_{i+1}t_{i+2}\dots t_{j-1}t_{i+1}t_{i+3}t_{j-2}]$ This is in turn equal to $t_i[p_i t_{i+2}\dots t_{j-1}t_{i+1}t_{i+3}t_{j-2}]$, by the relation $t_i p_i = t_i p_{i+1}$. We can then commute the p_i to the far right, yielding $xp_j = t_i t_{i+2}\dots t_{j-1}t_{i+1}t_{i+3}t_{j-2}p_i$ as desired. As discussed at the end of (i), one can easily prove that $xp_i = xp_i p_j$.

(iv) This proof is analogous to that of the previous case. □

4.3.4 Putting an order on P_n

By putting an ordering on P_n we can measure how close a given word is to being in RTL form and thereby better describe the actions necessary to perform our decomposition and get the "dead ends", or non-incident vertices on the right, and incident vertices on the left.

Definition 13. Let $<$ be any order on the power set of $\{1, \dots, n\}$ with the properties:
• If $X \subseteq Y$ then $X \leq Y$
• If $i, i+1 \notin X$ then $X \cup \{i\} < X \cup \{i+1\}$

Definition 14. A word $d \in M'_n$ is said to be in *minimal form* if it is in the form RTL for some $R \in RP_n, T \in TL_n(x)$ and $L \in LP_n$ with the property that for all $R' \in RP_n, T' \in TL_n(x)$, and $L' \in LP_n$ such that $d = R'T'L'$

$$\tau(R) \leq \tau(R') \text{ and } \beta(L) \leq \beta(L').$$

In this case, R and L are said to be *minimal*.

Theorem 5 (ordering on P)**.** *If $P \in P_n$ and $i \in \{1, \dots, n-1\}$ then $\tau(l_i P) \leq \tau(P)$ with equality if and only if $i, i+1 \notin \tau(P)$.*

Proof. Notice that if $i+1 \in \tau(P)$, left multiplication by r_i will send the top of P to the ith index, leaving $\tau(l_i P)$ empty in the index i on top, thus $\tau(l_i P) < L$, hence $r_i P < P$. Now if $j+1 \in \tau(P)$, then $r_j P < P$. Furthermore, if $j \in \tau(P)$, then left multiplication by r_j leaves the jth vertex on top of $r_j P$ empty, hence $r_j P < P$. Left multiplication by r_j only affects these two top vertices, thus in all other cases P is unaffected. Hence if $j, j+1 \notin P$, then $r_j P = P$, giving us the desired result. □

Theorem 6. *If $d = P_1 T P_2$ is minimal, then the following three assertions hold:*
(1) For all vertices i,
If vertex i' is connected to j' in T, then $i' \in \tau(P_2)$ if and only if $j' \in \tau(P_2)$.

If vertex i is connected to j' in T, then $i \in \beta(P_1)$ if and only if $j' \in \tau(P_2)$.
If vertex i is connected to j in T, then $i \in \beta(P_1)$ if and only if $j \in \beta(P_1)$.
(2) There exists a natural number k such that $\tau(P_2) = \{1, 2, \ldots, k\}$
(3) There exists a natural number m such that $\beta(P_1) = \{1, 2, \ldots, m\}$

Proof. (1) If any of the implications are false, then the hypothesis for at least one of the cases of lemma 7 are satisfied, which implies that d is not in minimal form.

(2) If $\tau(P_2) \neq \{1, \ldots, k\}$, for any k, there exists an i such that $i \notin \tau(P_2)$ and $i + 1 \in \tau(P_2)$. There are five ways in which this can happen. In each case we can prove that $P_1TP_2 = P_1'T'P_2'$, where $P_1' \leq P_1$ and $P_2' < P_2$, implying that P_1TP_2 was not minimal.

Case 1. i' is connected to j' with $i < j$.
Then $T = T_1(t_{j-1}t_{j-3}\cdots t_i)(t_{j-2}t_{j-4}\cdots t_{i+1})T_2$ where $T_1 \in \langle t_1, \ldots t_{n-1}\rangle, T_2 \in \langle t_{i+1} \ldots t_{j-2}\rangle$. Consider Te_i. We commute every Temperley-Lieb generator in T_2 with e_i. With the e_i in place, notice that the hypothesis for theorem 3 are satisfied where k in theorem 3 is equal to $j - 1$. We hop the dead end at i to $j - 1$. This yields:

$$\begin{aligned} Te_i &= T_1(t_{j-1}t_{j-3}\cdots t_i)(t_{j-2}t_{j-4}\cdots t_{i+1})T_2e_i \\ &= T_1(t_{j-1}t_{j-3}\cdots t_i)(t_{j-2}t_{j-4}\cdots t_{i+1})e_iT_2 \\ &= T_1(t_{j-1}t_{j-3}\cdots t_i)(t_{j-2}t_{j-4}\ldots t_{i+1})(t_{j-3}t_{j-5}\ldots t_i)(l_{j-2}l_{j-3}\ldots l_i)T_2. \end{aligned}$$

Subsequently, we slide (lemma 4) to get:

$$Te_i = T_1(t_{j-1}t_{j-3}\cdots t_i)(t_{j-2}t_{j-4}\ldots t_{i+1})(t_{j-3}t_{j-5}\ldots t_i)T_2'(l_{j-2}l_{j-3}\ldots l_i)$$

where $T_2' \in \langle t_i \ldots t_{j-3}\rangle$.
In conclusion,

$$d = P_1TP_2 = P_1T(l_{j-2}l_{j-3}\ldots l_i)P_2$$

and by theorem 5 $\tau((l_{j-2}l_{j-3}\ldots l_i)P_2) < \tau(P_2)$ since $i + 1 \notin P_2$

Case 2. i' is connected to j' with $i > j$.
In this case we have

$$T = T_1(t_{i-1}t_{i-3}\cdots t_j)(t_{i-2}t_{i-4}\cdots t_{j+1})T_2$$

where $T_1 \in \langle t_1, \ldots t_{n-1}\rangle$ and $T_2 \in \langle t_{j+1} \ldots t_{i-2}\rangle$. Since $i \notin \tau(P_2)$ we have $d = P_1Tp_iP_2 = P_1Tp_ip_jP_2$ (by Lemma 7). We can now obtain P_2' such that $\tau(P_2') < \tau(P_2)$ by referring to the previous case.

Case 3. i' is connected to j with $j < i$
In the case, T is of the form

$$T = T_1(t_{j+1}t_{j+3}\cdots t_{i-1})(t_jt_{j+2}\cdots t_{i-2})T_2$$

where $T_1 \in \langle t_{j+1}, \dots t_{n-1}\rangle$ and $T_2 \in \langle t_1 \dots t_{i-2}\rangle$. Consider Tp_i. As in case 1 we can commute p_i through T_2. Notice that $j < i, i-j$ is even, and we have a divisor $x = (t_{j+1}t_{j+3}\cdots t_{i-1})(t_j t_{j+2}\cdots t_{i-2})$. Thus by lemma 7, we get:

$$\begin{aligned} Tp_i &= T_1(t_{j+1}t_{j+3}\cdots t_{i-1})(t_j t_{j+2}\cdots t_{i-2})T_2 p_i \\ &= T_1(t_{j+1}t_{j+3}\cdots t_{i-1})(t_j t_{j+2}\cdots t_{i-2})p_i T_2 \\ &= T_1 p_j (t_{j+1}t_{j+3}\cdots t_{i-1})(t_j t_{j+2}\cdots t_{i-2})T_2 \end{aligned}$$

Observe that taking the antiisomorphism of theorem 3 (hop) we obtain:

$$\begin{aligned} [(t_{k-1}t_{k-3}\dots t_{i+1})p_i]^* &= [(t_{k-1}t_{k-3}\dots t_{i+1})(t_{k-2}t_{k-4}\dots t_i)(l_{k-1}l_{k-2}\dots l_i)]^* \\ p_i(t_{i+1}t_{i+3}\dots t_{k-1}) &= (r_i r_{i+1}\dots r_{k-1})(t_i t_{i+2}\dots t_{k-2})(t_{i+1}t_{i+3}\dots t_{k-1}) \end{aligned}$$

Hence by applying theorem 3,

$$\begin{aligned} Tp_i &= T_1 p_j (t_{j+1}t_{j+3}\cdots t_{i-1})(t_j t_{j+2}\cdots t_{i-2})T_2 \\ &= T_1(r_j r_{j+1}\dots r_{i-1})(t_j t_{j+2}\dots t_{i-2})(t_{j+1}t_{j+3}\dots t_{i-1})(t_j t_{j+2}\cdots t_{i-2})T_2. \end{aligned}$$

As in case 1 we can use the relation $t_a t_k t_a = t_a$ if $|a-k| = 1$ to simplify the product of t's. Now we use the relation $r_k = r_k p_{k+1}$ and apply Lemma 6 to produce all the r's between j and $n-1$. So,

$$\begin{aligned} &= T_1(r_j r_{j+1}\dots r_{i-1})(t_j t_{j+2}\cdots t_{i-2})T_2 \\ &= T_1(r_j r_{j+1}\dots r_{i-1})p_i(t_j t_{j+2}\cdots t_{i-2})T_2 \\ &= T_1(r_j r_{j+1}\dots r_{i-1})r_i l_i(t_j t_{j+2}\cdots t_{i-2})T_2 \\ &= T_1(r_j r_{j+1}\dots r_{i-1})(r_i\dots r_{n-1})(l_{n-1}\dots l_i)(t_j t_{j+2}\cdots t_{i-2})T_2 \\ &= T_1(r_j r_{j+1}\dots r_{i-1})(r_i\cdots r_{n-1})(t_j t_{j+2}\cdots t_{i-2})T_2(l_{n-1}\cdots l_i). \end{aligned}$$

Now the hypothesis for slide (theorem 4) are satisfied, so we obtain:

$$Tp_i = (r_i\cdots r_{n-1})T_1'(t_j t_{j+2}\cdots t_{i-2}T_2(l_{n-1}\cdots l_i)$$

where $T_1' \in \langle t_2 \dots t_{i-1}\rangle$. In conclusion,

$$d = P_1 T P_2 = P_1 T p_i P_2 = P_1(r_i\cdots r_{n-1})T_1'(t_j t_{j+2}\cdots t_{i-2}T_2(l_{n-1}\cdots l_i)P_2$$

. Notice that $\beta(P_1 r_i\cdots r_{n-1}) \le \beta(P_1)$ and $\tau(l_{n-1}\cdots l_i P_2) < \tau(P_2)$ by theorem 5, since $i+1 \notin \tau(P_2)$.

Case 4. i' is connected to j with $j > i$
In this case T can be written in the form $T_1(t_{j-2}t_{j-4}\cdots t_i)(t_{j-1}t_{j-3}\cdots t_{i+1})T_2$ where $T_1 \in \langle t_1, \dots t_{j-2}\rangle, T_2 \in \langle t_{i+1}\dots t_{n-1}\rangle$. Consider Tp_i. Every element in T_2 commutes with p_i. Notice that the hypothesis in theorem 3 are satisfied for $k = j$. Thus,

$$\begin{aligned} Tp_i &= T_1(t_{j-2}t_{j-4}\cdots t_i)(t_{j-1}t_{j-3}\cdots t_{i+1})T_2 p_i \\ &= T_1(t_{j-2}t_{j-4}\cdots t_i)(t_{j-1}t_{j-3}\cdots t_{i+1})p_i T_2 \\ &= T_1(t_{j-2}t_{j-4}\cdots t_i)(t_{j-1}t_{j-3}\cdots t_{i+1})(t_{j-2}t_{j-4}\cdots t_i)(l_{j-1}l_{j-2}\cdots l_i)T_2. \end{aligned}$$

Using the relations $t_i t_j t_i = t_i$ if $|i-j| = 1$ and $t_i t_j = t_j t_i$ if $|i-j| \geq 2$ as in case 1, we can simplify the product of t's to $(t_i t_{i+2} \cdots t_{j-2})$. Now we use the relation $r_i = p_{i+1} l_i$ and apply wallslide (Lemma 6) to produce all the l's between j and $n-1$, so that we will be able to slide the product of l's (theorem 4) through T_2. So we have:

$$\begin{aligned}
&= T_1(t_{j-2}t_{j-4}\cdots t_i)(l_{j-1}l_{j-2}\cdots l_i)T_2 \\
&= T_1(t_{j-2}t_{j-4}\cdots t_i)p_j(l_{j-1}l_{j-2}\cdots l_i)T_2 \\
&= r_jT_1(t_{j-2}t_{j-4}\cdots t_i)(l_jl_{j-1}l_{j-2}\cdots l_i)T_2 \\
&= r_jT_1(t_{j-2}t_{j-4}\cdots t_i)p_{j+1}(l_jl_{j-1}l_{j-2}\cdots l_i)T_2 \\
&= r_jr_{j+1}T_1(t_{j-2}t_{j-4}\cdots t_i)(l_{j+1}l_jl_{j-1}l_{j-2}\cdots l_i)T_2
\end{aligned}$$

We can continue this process until we arrive at:
$(r_jr_{j+1}\cdots r_{n-2}r_{n-1})T_1(t_{j-2}t_{j-4}\cdots t_i)(l_{n-1}l_{n-2}\cdots l_jl_{j-1}l_{j-2}\cdots l_i)T_2$

Now the hypotheses for slide (lemma 4) are satisfied, we obtain:

$$Tp_i = (r_{n-1}r_{n-2}\cdots r_j)T_1(t_it_{i+2}\cdots t_{j-2})T_2'(l_{n-1}l_{n-2}\cdots l_i)$$

where $T_2' \in \langle t_i \ldots t_{n-2}\rangle$.

In conclusion,

$$d = P_1Tp_iP_2 = P_1(r_{n-1}r_{n-2}\cdots r_j)T_1(t_it_{i+2}\cdots t_{j-2})T_2'(l_{n-1}l_{n-2}\cdots l_i)P_2$$

. Notice that $\tau(P_1r_{n-1}r_{n-2}\cdots r_j) \leq \tau(P_1)$ and that $\tau((l_{n-1}l_{n-2}\cdots l_i)P_2) < \tau(P_2)$, implying that the from P_1TP_2 was not minimal.

Case 5: i' is connected to j on top with $j = i$
T can be written as $T = T_1T_2$ where $T_1 \in \langle t_1, t_2, \ldots, t_{i-2}\rangle$, $T_2 \in \langle t_{i+1}, t_{i+2}, \ldots, t_{n-1}\rangle$

The proof for this case is nearly identical to that of Case 4. We apply the hop relation to p_i to get it through t_{i+1} if it appears in the product (otherwise we perform a wallslide), then produce r_i and l_i and commute r_i through T_1, and furthermore, obtain some product of l's and apply theorem 4 to slide them through T_2.

Case 2: If T_2 does not contain any factors of t_{i+1}, we apply wallslide (Lemma 6) to get $Tp_i = r_iT_2l_i = r_iTl_i$. Note that we can commute the r_i through T_2 since $T_2 \in \langle t_{i+2}, \ldots, t_{n-1}\rangle$, and $T_1 \in \langle t_1, \ldots t_{i-2}\rangle$ Thus $d = P_1TP_2 = P_1r_iTl_iP_2$ and $\tau(P_1r_i) \leq \tau(P_1)$ and $\tau(l_iP_1) < \tau(P_1)$, by theorem 5 since $i+1 \in \tau(L)$

If (3) does not hold, we can prove that d is not minimal using analogous reasoning.

□

4.3.5 Standard Form

In this section we show theorem 7, that given a word $d \in M'_n$ written in the form RTL, where R and L are minimal, d can be written as a word in standard form. Note that the difference between RTL and standard form concerns edges in T whose endpoints are empty vertices of R or L. The following lemma covers an extreme case.

Lemma 8. *Let T be an arbitrary element of TL_n composed of generators whose indices are greater than or equal to some natural number k, with $k < n$. Let $E = \prod_{i=k}^{n} p_i$. Then*

$$E \cdot T \cdot E = E.$$

Proof. We proceed by induction on the number of letters in T. First consider the base case, when $T = t_i$ for some i between k and $n-1$. Then we obtain $ETE = Ep_it_ip_iE = E$ by the relations $p_i^2 = p_i, p_ip_j = p_jp_i$ and $p_it_ip_i = p_ip_{i+1}$ (from relation 9).

Assume that $ETE = E$ whenever T is the product of m letters. We wish to show that $ETt_iE = ETE$ for any i greater than or equal to k, thereby proving that $ETE = E$ whenever T is the product of $m+1$ letters. Consider the following cases for what the i'th vertex of T is connected to to.

Case 1: i' is connected to j' with $i' < j' - 1$. Then T is of the form

$$T = T_1(t_it_{i+2}\cdots t_{j-1})(t_{i+1}t_{i+3}\cdots t_{j-2})T_2$$

where $T_1 \in \langle t_1, \dots t_{n-1}\rangle, T_2 \in \langle t_{i+1} \dots t_{j-2}\rangle$.

$$\begin{aligned}
ETt_iE &= ET_1(t_it_{i+2}\cdots t_{j-1})(t_{i+1}t_{i+3}\cdots t_{j-2})T_2t_iE \\
&= ET_1(t_it_{i+2}\cdots t_{j-1})(t_{i+1}t_{i+3}\cdots t_{j-2})p_jT_2t_iE \text{ (by } p_i^2 = p_i, p_ip_j = p_jp_i) \\
&= ET_1(t_it_{i+2}\cdots t_{j-1})(t_{i+1}t_{i+3}\cdots t_{j-2})p_jp_iT_2t_iE \text{ (by theorem 7 part (iii))} \\
&= ET_1(t_it_{i+2}\cdots t_{j-1})(t_{i+1}t_{i+3}\cdots t_{j-2})p_jT_2p_it_ip_iE \text{ (} p_i \text{ commutes with any } t \text{ in } T_2) \\
&= ET_1(t_it_{i+2}\cdots t_{j-1})(t_{i+1}t_{i+3}\cdots t_{j-2})p_jT_2E \text{ (since } p_it_ip_i = p_i) \\
&= ETE = E \text{ (by the inductive hypothesis).}
\end{aligned}$$

Case 2: i' is connected to j' with $k < j' < i'$.
Then T is of the form $T_1(t_jt_{j+2}\cdots t_{i-1})(t_{j+1}t_{j+3}\cdots t_{i-2})T_2$, where $T_1 \in \langle t_1, \dots t_{n-1}\rangle, T_2 \in \langle t_{j+1} \dots t_{i-2}\rangle$.

$$\begin{aligned}
ETt_iE &= ET_1(t_jt_{j+2}\cdots t_{i-1})(t_{j+1}t_{j+3}\cdots t_{i-2})T_2t_iE \\
&= ET_1(t_jt_{j+2}\cdots t_{i-1})(t_{j+1}t_{j+3}\cdots t_{i-2})T_2t_ip_jE \text{ (by } p_j^2 = p_j, p_ip_j = p_jp_i) \\
&= ET_1(t_jt_{j+2}\cdots t_{i-1})(t_{j+1}t_{j+3}\cdots t_{i-2})p_jT_2t_iE \text{ (by commuting)} \\
&= ET_1(t_jt_{j+2}\cdots t_{i-1})(t_{j+1}t_{j+3}\cdots t_{i-2})p_iT_2t_iE \text{ (by theorem 7 part (iv))} \\
&= ET_1(t_jt_{j+2}\cdots t_{i-1})(t_{j+1}t_{j+3}\cdots t_{i-2})T_2p_it_ip_iE \text{ (by } p_i^2 = p_i\text{, commuting)} \\
&= ET_1(t_jt_{j+2}\cdots t_{i-1})(t_{j+1}t_{j+3}\cdots t_{i-2})T_2E \text{ (since } p_it_ip_i = p_i) \\
&= ETE = E
\end{aligned}$$

Case 3: i' is connected to j, with $i' < j$. Then T is of the form $T_1(t_it_{i+2}\cdots t_{j-2})(t_{i+1}t_{i+3}\cdots t_{j-1})T_2$ where $T_1 \in \langle t_1, \ldots t_{j-2}\rangle, T_2 \in \langle t_{i+1}\ldots t_{n-1}\rangle$

$$\begin{aligned}
ETt_iE &= ET_1(t_it_{i+2}\cdots t_{j-2})(t_{i+1}t_{i+3}\cdots t_{j-1})T_2t_iE \\
&= ET_1p_j(t_it_{i+2}\cdots t_{j-2})(t_{i+1}t_{i+3}\cdots t_{j-1})T_2t_iE \text{ (by } p_j^2 = p_j\text{, commuting)} \\
&= ET_1(t_it_{i+2}\cdots t_{j-2})(t_{i+1}t_{i+3}\cdots t_{j-1})p_iT_2t_iE \text{ (by theorem 7 part (i))} \\
&= ET_1(t_it_{i+2}\cdots t_{j-2})(t_{i+1}t_{i+3}\cdots t_{j-1})T_2p_it_ip_iE \text{ (by } p_i^2 = p_i\text{, commuting)} \\
&= ET_1(t_it_{i+2}\cdots t_{j-2})(t_{i+1}t_{i+3}\cdots t_{j-1})T_2E \text{ (since } p_it_ip_i = p_i) \\
&= ETE = E
\end{aligned}$$

Case 4: i' is connected to j, with $j < i'$. The proof of this case is analogous to Case 3.

Case 5. i' is connected to i. This implies that T does not contain t_i or t_{i-1}. Thus we obtain $ETt_iE = ETp_it_ip_iE = ETE = E$, since $p_it_{i+1} = t_{i+1}p_i$.

Case 6: i' is connected to $i'+1$. This implies by section 1 that $T = T't_i$, for some T'. Thus we obtain

$$ETt_iE = ET't_i^2E = ET't_iE = ETE = E$$

□

If a Temperley-Lieb diagram is multiplied on bottom by a planar rook diagram whose vertices are all empty past a certain vertex k, and all vertices on bottom with index greater than or equal to k are connected only to each other, then it does not matter what the bottom of the Temperley-Lieb diagram looks like past k. This fact about diagrams motivates the following lemma.

Lemma 9. $(t_it_{i+2}\cdots t_k)(p_{k+2}p_{k+3}\cdots p_n)T(p_ip_{i+1}\cdots p_n) = (t_it_{i+2}\cdots t_k)(p_ip_{i+1}\cdots p_n)$, *where* $T \in \langle t_i, \ldots, t_{n-1}\rangle$.

Proof. Recall the relation $t_jp_jt_j = t_jp_jp_{j+1}t_j = t_j$. We can apply this relation to get:

$$\begin{aligned}
& (t_it_{i+2}\cdots t_k)(p_{k+2}p_{k+3}\cdots p_n)T(p_ip_{i+1}\cdots p_n) \\
= \; & (t_it_{i+2}\cdots t_k)(p_ip_{i+1}\cdots p_{k+1})(t_it_{i+2}\cdots t_k)(p_{k+2}p_{k+3}\cdots p_n)T(p_ip_{i+1}\cdots p_n) \\
= \; & (t_it_{i+2}\cdots t_k)(p_ip_{i+1}\cdots p_n)(t_it_{i+2}\cdots t_k)T(p_ip_{i+1}\cdots p_n) \\
= \; & (t_it_{i+2}\cdots t_k)(p_ip_{i+1}\cdots p_n) \text{ (by 8)}
\end{aligned}$$

□

Theorem 7. *Let d be a word in M_n' in the form RTL where R and L are minimal. Then d can be put into standard form.*

Proof. We consider two cases.

Case 1: $|\beta(R)| \neq |\tau(L)|$ Assume without loss of generality that $|\tau(L)| < |\beta(R)|$. This means that no vertex on bottom with index greater than $|\tau(L)|$ can be connected to a vertex on bottom whose index is less than or equal to $|\tau(L)|$ or any vertex on top with index less than or equal to $|\beta(R)|$. Similarly, no vertex on top with index greater than $|\beta(R)|$ can be connected to a vertex on top whose index is less than or equal to $|\beta(R)|$ or any vertex on bottom with index less than or equal to $|\tau(L)|$. (If this were not the case, L or R would not be minimal, as proven in theorem 6.) By appealing to Temperley-Lieb diagrammatic intuition, this means that T can be written in the form $T_a t_{|\tau(L)|+1} t_{|\tau(L)|+3} \cdots t_{|\beta(R)|-1} T_b$, where $T_a \in \langle t_1, t_2, \ldots, t_{|\beta(R)|-1} \rangle$ and $T_b \in \langle t_{|\tau(L)|}, t_{|\tau(L)|+1}, \ldots, t_{n-1} \rangle$.

The word d can be written in the form

$$d = R' p_{|\beta(R)|+1} p_{|\beta(R)|+2} \cdots p_n T_a t_{|\tau(L)|+1} t_{|\tau(L)|+3} \cdots t_{|\beta(R)|-1} T_b p_{|\tau(L)|+1} p_{|\tau(L)|+2} \cdots p_n L'$$

for some $R' \in RP_n$ and some $L' \in LP_n$.

$$\begin{aligned} d &= R' T_a t_{|\tau(L)|+1} t_{|\tau(L)|+3} \cdots t_{|\beta(R)|-1} p_{|\beta(R)|+1} p_{|\beta(R)|+2} \cdots p_n T_b p_{|\tau(L)|+1} p_{|\tau(L)|+2} \cdots p_n L' \\ &= R' T_a t_{|\tau(L)|+1} t_{|\tau(L)|+3} \cdots t_{|\beta(R)|-1} p_{|\tau(L)|+1} p_{|\tau(L)|+2} \cdots p_n L' \text{ (by lemma 8)} \end{aligned}$$

which is in standard form.

Case 2: $|\beta(R)| = |\tau(L)| = k$

Using similar reasoning as above, no vertex whose index is greater than or equal to k can be connected to any vertex less than k. This implies that T can be written without any factors of t_{k-1}. By far commutativity, d can be written in the from $RT_a p_k p_{k+1} \cdots p_n T_b p_k p_{k+1} \cdots p_n L$, where $T_a \in \langle t_1, t_2, \ldots t_{k-2} \rangle$, and $T_b \in \langle t_k t_{k+1}, \ldots, t_{n-1} \rangle$, which is equal to $RT_a p_k p_{k+1} \cdots p_n L$ by 8, which in standard form. □

Theorem 8. *Every word in M_n' is equivalent to a standard word in M_n.*

Proof. We proceed by induction. Note that every single-letter word is in standard form. Assume that every word of k letters is equivalent to a word in standard form. Let d be an arbitrary word of $k+1$ letters. Let d_0 be the product of the first k letters of d, and let x be the final letter of d. By the inductive hypothesis, d_0 is equivalent to a standard word. If $x \in P_n$, then $d_0 x$ is already in PTP form. If $x \in TL_n$, then, by Lemma 2, $d_0 x$ can be put into PTP form. Thus, d is equivalent to a word in PTP from. By theorems 6 and 7, this implies that d is equivalent to a word in standard from. Thus every string of $k+1$ letters is equivalent to a word in standard form.

Thus every word in M_n' is equivalent to a word in standard from. □

Theorem 9. *The monoid M_n' is isomorphic to M_n.*

Proof. By theorem 8, we know that the number of distinct words in M_n' is equal to the number of standard words. Clearly there is a one-to-one correspondence between words in standard form and diagrams in M_n which are in standard form. Every diagram in M_n is equivalent to a diagram in standard from. Thus $|M_n'| = |M_n|$. This implies that the homomorphism ϕ is an isomorphism. □

5 Conclusion

In this paper we have introduced the rook, planar rook, left and right planar rook monoids, the Temperley-Lieb algebra, and the Motzkin Monoid. We have given a complete presentation of the Motzkin Monoid through a set of moves (hop, burrow, wallslide), whose proof at times can get very involved. Monoid presentations are very important in order for us to understand a monoid, and in order for us to define morphisms to other monoids. The Motzkin Monoid is used extensively in [4]. Another example of a presentation is completed in [10]. In this paper, David Penneys gives a presentation of the Annular Temperley-Lieb category.

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