MAPPING BRAIDS INTO COLORED PLANAR ROOK DIAGRAMS

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September 15, 2014

Contents

Acknowledgements

The mathematical content in this thesis is derived from a collaboration over the summer during UCSB 2012 REU, with Anthony Caine (Northern Arizona University). Due to the nature of the REU setting, most of this work was done collaboratively, however, the addition of the the Mathematical Background and Motivation chapter, the structure of the paper, and many details and adaptations of the proofs are my own work.

The research for this thesis was support by the National Science Foundation Research Experience for Undergraduates Program (NSF-REU Grant DMS-**********). *look up grant number*

I would particularly like to thank my advisor, Stephen Bigelow.

Abstract

Braid groups are well known and have been extensively studied since their introduction by Artin in the 1920s. Artin introduced the braid group in order to provide an algebraic approach to the study of knots. In this paper, we utilize the planar rook algebra, an algebra consisting of diagrams that resemble braids, and has a well understood representation theory (see [9]). We define a non-trivial homomorphism, $\phi : B_n \to (\mathbb{C}P_{n,2})^*$, that maps the braid group into the group of units of $\mathbb{C}P_{n,2}$, where $\mathbb{C}P_{n,2}$ is the algebra of formal linear combinations of two-colored planar rook diagrams. We define a trace, $\mathrm{Tr}_n : \mathbb{C}B_n \to \mathbb{C}$, from the braid algebra to the complex numbers, through $\mathbb{C}P_{n,2}$, and we show our trace is invariant under Markov moves. From Alexander [1] and [2], knowing every oriented link is isotopic to a closed braid and two closed braids are equal if and only if one can be attained from the other through a finite sequence of Markov Moves (by [7]), Tr_n can be used to define an invariant of knots and links. In fact, $\mathrm{Tr}_n(\hat{x})$ is the Jones Polynomial of the closed braid $\hat{x}$ up to some renormalization.

1. Mathematical Background and Motivation

1.1. The Braid Group.

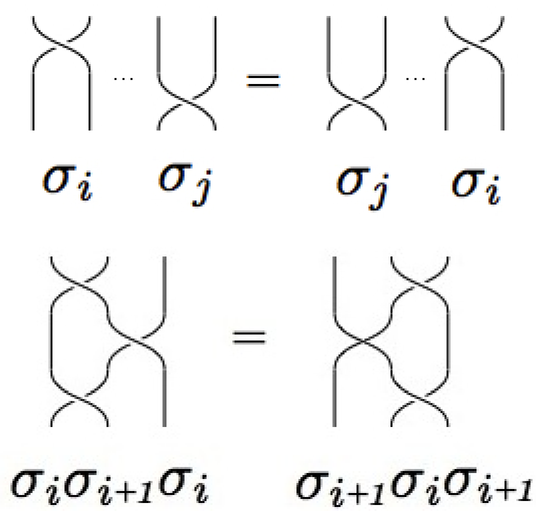
Definition 1.1. *The braid group, denoted B_n, is the group generated by $\sigma_1, \ldots, \sigma_{n-1}$, modulo the relations:*

(1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ *for* $|i-j| > 1$
(2) $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_j \sigma_i$ *for* $|i-j| = 1$

These relations are better motivated in the context of a geometric definition of braids.

Definition 1.2. *A braid, b is an arrangement of n monotonic strings in space up to isotopy from n fixed nodes on top to n fixed nodes on bottom. We define the binary $*$ operation $b_1 * b_2$, for braids b_1, b_2 as the result of identifying the bottom nodes of b_1 to the corresponding top nodes of b_2. The collection of all braids b on n strings is denoted by B_n.*

Through this definition, B_n with the operation $*$ is a group. Now, we define σ_i by braiding the i^{th} strand over the $i+1^{th}$ and σ_i^{-1} by braiding the i^{th} strand under the $i+1^{th}$. Using this approach, we can represent the relations in our algebraic definition as below:



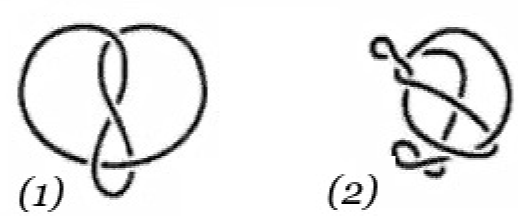
1.2. Knots and Links.

The next two definitions are of topological objects known as knots and links.

Definition 1.3. *A knot is a smooth closed curve in space that does not intersect itself anywhere. Two knots are considered to be the same if we can deform the one knot to the other knot without ever having it intersect itself.*

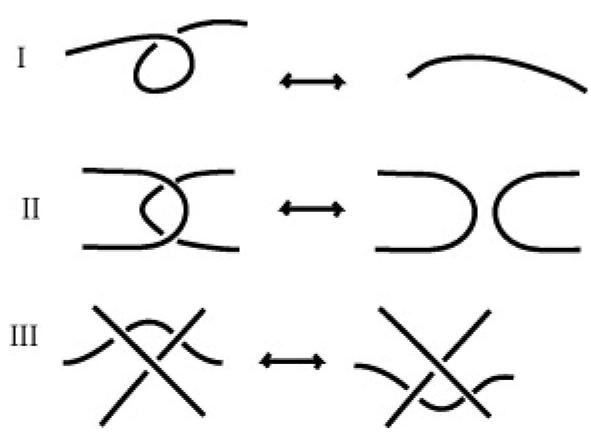
Definition 1.4. *A link is a set of knotted loops all tanged up together. Two links are considered to be the same if we can deform the one link to the other link without ever having any one of the loops intersect itself or any of the other loops in the process.*

In other words, knots and links can be seen as strings in 3-space modulo isotopy. Another way of visualizing them is through their projection into 2-space. Since every link is a collection of knots, we can focus our attention to just knots for the moment. In the following example, notice that (1) and (2) are different projections of the same knot:

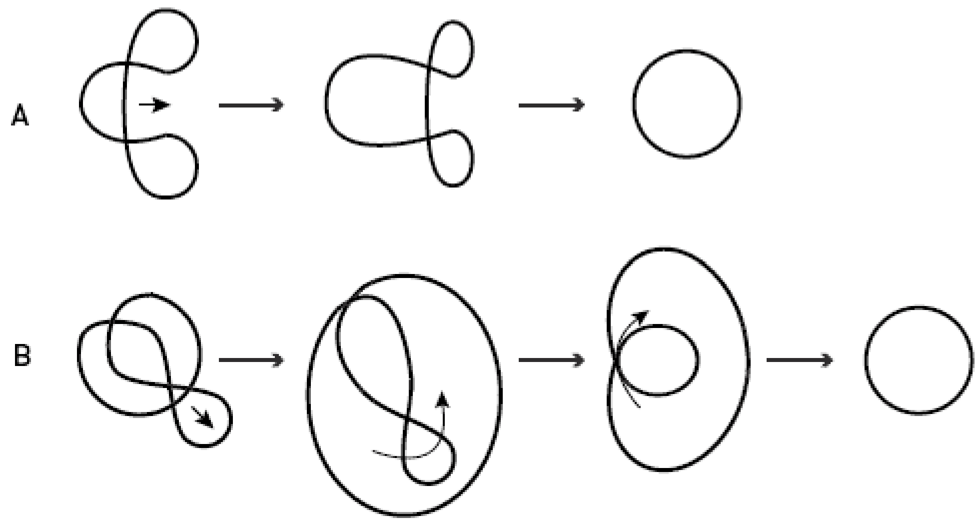


Two projections represent the same knot if and only if they are connected by Reidemeister moves, defined below:

Definition 1.5. *A Reidemeister move is one of three ways to change a projection of the knot that will change the relation between the crossings. The first allows us to put in or take out a twist in the knot. The second allows us to either add two crossings or remove two crossings. The third and final allows us to slide a strand of the knot from one side of a crossing to the other side of the crossing.*



We see that the following projections of knots are both the unknot through Reidemeister moves:



Since a knot can be represented by different knot projections, we introduce the concept of a knot invariant:

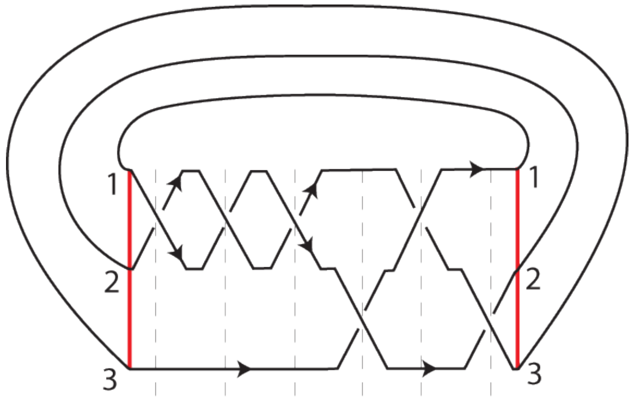
Definition 1.6. *A knot invariant is a function that assigns a knot projection a value and assigns the same value to knot projections that represent the same knot. If a knot invariant gives two different values for two different knot projections, then it tells you the projections represent two distinct knots.*

A specific form of a knot invariant is a knot polynomial:

Definition 1.7. *A knot polynomial is knot invariant that inputs a link projection and whose output is a Laurent polynomial.*

The next definition gives us a relationship between links and braids.

Definition 1.8. *Given a braid, b, its closure is the result of attaching each node on top with the corresponding node on bottom, as in the figure below. The 1^{st} node on top is attached with the 1^{st} on bottom and so on. We denote the closure of b by $\hat{b}$.*



Notice that the braid closure process turns a braid into a link. Alexander is credited with the following theorem (proof found in [1] and [2]):

Theorem 1.9. *Every oriented link is isotopic to a closed braid.*

This relationship is not a bijection, as two distinct braids can have isotopic closures. Markov explored this many-to-one relationship by creating equivalence classes of braids with the below relations, now known as "Markov moves"

(1) $ab \sim ba$
(2) $\iota(a)\sigma_n \sim a$
(3) $\iota(a)\sigma_n^{-1} \sim a$

where $a, b \in B_n$ and we define $\iota : B_n \to B_{n+1}$ to be the map that adds an additional strand connecting the top $n+1^{th}$ node to the bottom $n+1^{th}$ node. He then proved the following result (found in [7]):

Theorem 1.10. *Given two braids, b_1 and b_2, $\hat{b_1} = \hat{b_2}$ if and only if $b_1 \sim b_2$ where $\sim$ is defined by the relations above.*

Next we wish to define a function known as a Markov Trace. Before we define a Markov trace, we need to define other intermediate algebraic objects.

Definition 1.11. *An associative algebra over a field K is a K-vector space A together with a monoid structure such that the multiplicative map $A \times A \to A$ is bilinear. All of our associative algebras will be over the field $\mathbb{C}$.*

Definition 1.12. *A trace function, tr: $A \to F$ is a linear function from an algebra, A, to a field F that satisfies $tr(ab) = tr(ba)$ for all $a, b \in A$. In this paper, $A = \mathbb{C}P_{n,2}$.*

As an example, recall the trace function of a matrix, in which $\text{tr}(M) = \sum_{i=1}^{n} a_{ii}$, where M is in the set of $n \times n$ matrices over a field F and each a_{ii} is an diagonal entry in the matrix with $a_{ii} \in F$. We are now ready to define a Markov trace:

Definition 1.13. *A Markov trace is a trace function that is invariant under the Markov moves.*

In [6], a specific associative algebra called the Planar Rook Algebra, denoted $\mathbb{C}P_n$ is defined. Our interest in this paper is to extend this algebra to a "2" colored Planar Rook Algebra we denote by $\mathbb{C}P_{n,2}$. In order to define $\mathbb{C}P_{n,2}$, we first wish to define $P_{n,2}$, the set of planar rook diagrams on two colors.

1.3. **The Planar Rook Algebra.**

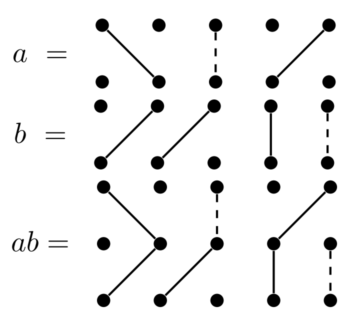
Definition 1.14. $P_{n,2}$ *is a monoid whose elements are all the possible diagram constructed in the following way:*

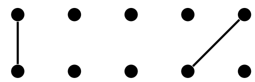
1. *We first start with a rectangle with n nodes on top, and n nodes on bottom.*
2. *We connect nodes by lines, known as "edges." Each edge connects a node at the top to a node at the bottom. Each node is an endpoint of either one or zero edges.*
3. *Each edge has one of two colors and we require that no two edges of the same color can intersect when drawn.*

Given $a, b \in P_{n,2}$*, our monoid operation is defined by an algorithm:*

1. *Identify the bottom row of vertices in a with the top row of vertices in b. (We now have a diagram with three rows of vertices).*
2. *Delete all edges that terminate at the middle row (i.e. connect do not connect to another edge).*
3. *Think of two connecting edges as one long edge and delete the middle row of vertices.*
4. *If we have a dotted edge and a non-dotted edge connected, then we delete the edges.*

For example, for $a, b \in P_{n,2}$:



= 

For convenience later in the paper we wish to define the following terms:

Definition 1.15. *Given $d \in P_{n,2}$, we say:*

(1) d has a k-edge *from i to j if there exists a k-colored line connecting the i^{th} node on bottom to the j^{th} node on top.*
(2) d has a vertical k-edge *at i if there exists a k-colored line from i to i.*

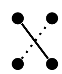
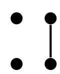
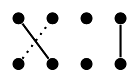
We now define $\mathbb{C}P_{n,2}$, to be the associative algebra generated by all formal linear combinations of diagrams in $P_{n,2}$ with coefficients in $\mathbb{C}$. The reason $\mathbb{C}P_{n,2}$ is of value to us is because of $\mathbb{C}P_n$'s simple presentation and the fact that its representations have been completely classified as seen in [9]. Since the representations of $\mathbb{C}P_n$ are well understood, we look for representations of B_n by first mapping B_n to $\mathbb{C}P_n$.

2. The Homomorphism

Before we discuss the function that results in our homomorphism, we must first establish important preliminary information.

2.1. **The Tensor Product.**

Definition 2.1. *For two diagrams $a \in P_{n_1,c}$ and $b \in P_{n_2,c}$ we define the tensor product of a and b to be $a \otimes b \in P_{n_1+n_2,c}$ where $a \otimes b$ is the diagram resulting from appending b to the right of a. For example,*

 $\otimes$  $=$  .

Note, the tensor product of a linear combination of diagrams is calculated by distributing and calculating the product term by term, i.e. for $A \in P_{n_1,c}$, $B \in P_{n_2,c}$, $C \in P_{n_3,c}$, $D \in P_{n_4,c}$ and $r, s, t, v \in \mathbb{C}$

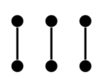
$$(rA + sB) \otimes (tC + vD) = rt(A \otimes C) + rv(A \otimes D) + st(B \otimes C) + sv(B \otimes D)$$

For simplicity, we wish to define further notation.

Definition 2.2. *Given a diagram $A \in P_{n,c}$, we define $A^{\otimes k}$ to be A concatenated with itself k times. For example, if*

$A =$ 

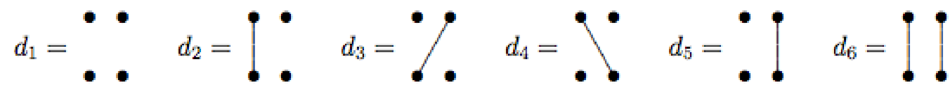
then, we have that for $k = 3$,

$A^{\otimes 3} =$ 

Now that we have that established, let us first understand the homomorphism of the 1-color diagrams, i.e. when $c = 1$ first defined in [6].

2.2. **The 1-color Homomorphism.**

To best understand generalizations that result later in the paper, we begin by observing a function $\phi : B_n \to (\mathbb{C}P_{n,1})^*$, where $(\mathbb{C}P_{n,1})^*$ is the group of units of $\mathbb{C}P_{n,1}$. Since the element of B_n can be viewed algebraically as words generated by permutations, without loss of generality, we can define our function from a single arbitrary permutation, $\sigma_i \in B_n$ by the multiplicative property of homomorphisms. Recalling the meaning of σ_i, it resembles the identity of B_n, except at nodes i and $i+1$. Thus, we wish to restrict our attention to the elements of $P_{n,1}$ that resemble the identity of $P_{n,1}$, but not at the ith and $i+1$st nodes. This leaves us with the elements of $P_{n,1}$ that resemble the identity of $P_{n,1}$, but have an element of $P_{2,1}$ at its ith and $i+1$st nodes. We enumerate $P_{2,1}$ below:



Now we have all of the necessary tools to define our function.

Definition 2.3. *Given $\sigma_i \in B_n$, we define $\phi : B_n \to (\mathbb{C}P_{n,1})^*$, where $(\mathbb{C}P_{n,1})^*$ is the group of units of $\mathbb{C}P_{n,1}$, on σ_i, as follows:*

$$\phi(\sigma_i) = a \cdot d_{1i} + b \cdot d_{2i} + c \cdot d_{3i} + d \cdot d_{4i} + e \cdot d_{5i} + f \cdot d_{6i}$$

where $a, b, c, d, e, f \in \mathbb{C}$, and

$$d_{ji} = I^{\otimes i-1} \otimes d_j \otimes I^{\otimes n-i-1}$$

with d_j corresponding to the above enumeration of $P_{2,1}$ and $I =$  *, the identity of $(\mathbb{C}P_{1,1})^*$.*

It was proved in [6] that for any choice of the coefficients $c, d \in \mathbb{C}$, $cd \neq 0$, if

$$\begin{gathered} a = 1 - c - d + cd \\ b = -1 \\ e = -cd \\ f = 1 \end{gathered}$$

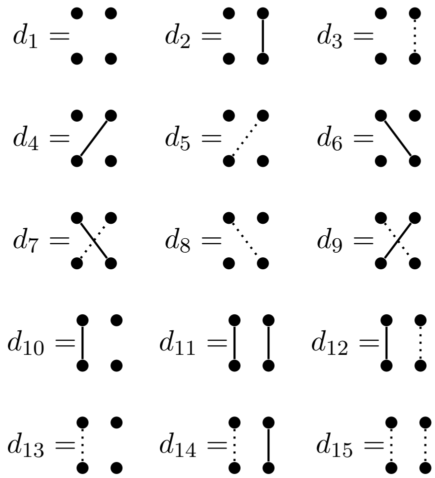
ϕ is in fact a homomorphism, and for σ_i^{-1} results in the following:

$$\phi(\sigma_i^{-1}) = (1 - \frac{1}{c} - \frac{1}{d} + \frac{1}{cd})d_{1i} - \frac{1}{cd}d_{2i} + \frac{1}{d}d_{3i} + \frac{1}{c}d_{4i} - d_{5i} + d_{6i}$$

Now, we wish to extend this result from the 1-color case to the 2-color case, using the same intuition.

2.3. **The 2-color Homomorphism.**

With the addition of a color, the way in which we define the homomorphism becomes slightly more burdensome. However, we will construct it in a similar fashion to the 1-color case. We start with an enumeration of $P_{2,2}$:



We can now define the 2-color function that we will use for our homomorphism.

Definition 2.4. *Given $\sigma_i \in B_n$, we define $\phi : B_n \to (\mathbb{C}P_{n,2})^*$, where $(\mathbb{C}P_{n,2})^*$ is the group of units of $\mathbb{C}P_{n,2}$, on σ_i, as follows:*

$$\phi(\sigma_i) = \sum_{j=1}^{15} t_j d_{ji}$$

where $t_j \in \mathbb{C}$, and

$$d_{ji} = I^{\otimes i-1} \otimes d_j \otimes I^{\otimes n-i-1}$$

with d_j corresponding to the above enumeration of $P_{2,2}$ and $I =$  *, the identity of $(\mathbb{C}P_{1,2})^*$.*

Theorem 2.5. *Given the function $\phi : B_n \to (\mathbb{C}P_{n,2})^*$ as defined above, and $q \in \mathbb{C}$ with $q \neq 0$, replacing the coefficients of ϕ, i.e., the t_j's as follows, makes ϕ a homomorphism:*

$$t_1 = t_2 = t_{15} = -q,$$
$$t_3 = t_9 = t_{13} = q$$
$$t_5 = t_8 = t_{14} = 0,$$
$$t_7 = t_{11} = 1,$$
$$t_4 = t_{12} = 1 - q,$$
$$t_6 = q - 1,\ t_{10} = q - 2$$

It is important to note that similar to [6], these coefficients are inspired by the Burau Representation and the proof of this theorem follows from the proof found in [6].

Corollary 2.6. *Defining $\phi(\sigma_i)$ with the above coefficients, we have that $\phi(\sigma_i^{-1})$ has the following coefficients:*

$$\begin{gathered}
t_1 = t_2 = -1, \\
t_3 = t_9 = t_{13} = 1, \\
t_4 = t_{12} = \tfrac{1}{q} - 1, \\
t_5 = t_8 = t_{14} = 0, \\
t_6 = 1 - \tfrac{1}{q}, \\
t_7 = t_{11} = \tfrac{1}{q}, \\
t_{10} = 1 - \tfrac{2}{q}, \\
t_{15} = -\tfrac{1}{q},
\end{gathered}$$

The proof follows from verifying

$$\phi(\sigma_i)\phi(\sigma_i^{-1}) = \phi(\sigma_i\sigma_i^{-1}) = \phi(\mathbb{1}) = I^{\otimes n}$$

where $\mathbb{1}$ is the empty word in B_n, and this relation was verified through use of mathematica.

3. The Jones Polynomial

Now we wish to put the previous chapter's results in the frame of knot theory by showing that our homomorphism can satisfy relations that result in the Jones Polynomial. However, we must lay the foundation in the next two sections. In particular, we begin with a very important subalgebra of $\mathbb{C}B_n$ seen in [12].

3.1. The Hecke Algebra.

Definition 3.1. *The Hecke Algebra, denoted $\mathcal{H}_n(q)$, is the quotient algebra of $\mathbb{C}B_n$ by the subalgebra generated by the following relation:*

$$(\sigma_i - 1)(\sigma_i + q) = 0 \quad i \in \{1, \ldots, n\}$$

Notice, distributing and multiplying by σ_i^{-1} gives us the equivalent relation:

$$\sigma_i - q\sigma_i^{-1} = (1-q)\mathbb{1} \quad i \in \{1, \ldots, n\}$$

This relation is above is known as a "skein relation", i.e. a linear combination of the braids σ_i, σ_i^{-1}, and $\mathbb{1}$, where $\mathbb{1}$ is the identity of B_n, i.e. the "empty word" in B_n. Recalling the homomorphism defined in the previous chapter, we conclude that our homomorphism preserves the above relation.

Lemma 3.2. $\phi : B_n \to (\mathbb{C}P_{n,2})^*$ *satisfies the following relation:*

$$\phi(x\sigma_i) - q\phi(x\sigma_i^{-1}) = (1-q)\phi(x)$$

for all $x \in B_n$, $i \in \{1, \ldots, n-1\}$.

Proof. Suppose $j \in \{1, \ldots, n-1\}$ and $x \in B_n$. Consider the following relation:

$$\phi(x\sigma_i) - q\phi(x\sigma_i^{-1}) = (1-q)\phi(x)$$

multiplying both sides of the relation by $\phi^{-1}(x)$ and distributing yields:

$$\begin{aligned} \phi^{-1}(x)(\phi(x\sigma_i) - q\phi(x\sigma_i^{-1})) &= \phi^{-1}(x)((1-q)\phi(x)) \\ \Rightarrow \phi^{-1}(x)\phi(x\sigma_i) - q\phi^{-1}(x)\phi(x\sigma_i^{-1}) &= (1-q)\phi^{-1}(x)\phi(x) \end{aligned}$$

Since ϕ is a homomorphism,

$$\begin{gathered}\phi^{-1}(x)\phi(x\sigma_i) - q\phi^{-1}(x)\phi(x\sigma_i^{-1}) = (1-q)\phi^{-1}(x)\phi(x)\\ \Rightarrow \phi(\sigma_i) - q\phi(\sigma_i^{-1}) = (1-q)\phi(\mathbb{1})\end{gathered}$$

Thus, it suffices to prove that

$$\phi(\sigma_i) - q\phi(\sigma_i^{-1}) = (1-q)\phi(\mathbb{1})$$

holds in order to prove the lemma. Note by definition of $\phi(\sigma_i)$ and $\phi(\sigma_i^{-1})$, we can gather the terms to have the following coefficients on the left side of the equations:

$$\begin{array}{ccc}
t_1 : -q - q(-1) & t_2 : -q - q(-1) & t_3 : q - q(1) \\
t_4 : 1 - q - q(\frac{1}{q} - 1) & t_5 : 0 - q(0) & t_6 : q - 1 - q(1 - \frac{1}{q}) \\
t_7 : 1 - q(\frac{1}{q}) & t_8 : 0 - q(0) & t_9 : q - q(1) \\
t_{10} : q - 2 - q(1 - \frac{2}{q}) & t_{11} : 1 - q(\frac{1}{q}) & t_{12} : 1 - q - q(\frac{1}{q} - 1) \\
t_{13} : q - q(1) & t_{14} : 0 - q(0) & t_{15} : -q - q(-\frac{1}{q})
\end{array}$$

which simplify to:

$$\begin{array}{ccc}
t_1 : 0 & t_2 : 0 & t_3 : 0 \\
t_4 : 0 & t_5 : 0 & t_6 : 0 \\
t_7 : 0 & t_8 : 0 & t_9 : 0 \\
t_{10} : 0 & t_{11} : 0 & t_{12} : 0 \\
t_{13} : 0 & t_{14} : 0 & t_{15} : 1 - q
\end{array}$$

Note, the coefficients of $\phi(\mathbb{1}) = I^{(\otimes n)}$ are,

$$\begin{array}{ccc}
t_1 : 0 & t_2 : 0 & t_3 : 0 \\
t_4 : 0 & t_5 : 0 & t_6 : 0 \\
t_7 : 0 & t_8 : 0 & t_9 : 0 \\
t_{10} : 0 & t_{11} : 0 & t_{12} : 0 \\
t_{13} : 0 & t_{14} : 0 & t_{15} : 1
\end{array}$$

and so we have

$$\phi(\sigma_i) - q\phi(\sigma_i^{-1}) = (1-q)\phi(\mathbb{1})$$

and we arrive at the desired conclusion. $\square$

From our definition of the Hecke Algebra and the above lemma, we obtain the immediate result:

Corollary 3.3. *All representations of B_n through ϕ are representations of $\mathcal{H}_n(q)$.*

3.2. **Our Markov Trace.**

Before we can define our Markov Trace, consider the following definition motivated by [6].

Definition 3.4. *For any $\beta \in \mathbb{C}$, the bubble trace function, $tr_n^\beta : \mathbb{C}P_{n,2} \to \mathbb{C}$ is the linear function that acts on diagrams, $d \in P_{n,2}$, by $tr_n^\beta(d) = \beta^{k_2(d)}$, where $k_2(d)$ is the number of 2-vertical lines in d.*

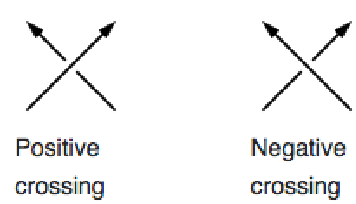
We must prove this is a trace.

Lemma 3.5. *For all diagrams $a, b \in P_{n,2}$, $tr^\beta(ab) = tr_n^\beta(ba)$.*

Proof. Suppose $a, b \in P_{n,2}$ and consider the function tr_n^β defined above. Note, it is sufficient to prove that $k_2(ab) = k_2(ba)$, i.e., ab and ba have the same number of 2-vertical lines. Moreover, this implies we can restrict our attention to the different ways a vertical line can be formed in a product. Our first way of achieving a 2-vertical line is the concatenation of two 2-vertical lines. Since the concatenation of two 2=vertical lines will be the same in either order, we see this case will be preserved by our trace. Now, the only other way to achieve a 2-vertical line is by a 2-edge from i to j in a and a 2-edge from j to i in b, with $j \neq i$. Taking the product in either order results in a 2-vertical line, whose position has not been preserved. However, the existence is what we care about, so therefore, $k_2(ab) = k_2(ba)$ and the result follows. $\square$

Now we require one final definition before we can propose a trace that we will verify is a Markov trace.

Definition 3.6. *The writhe of a link x, $w(x)$, is the total number of positive crossings minus the total number or negative crossings. Where the distinction between these two crossings can be seen below:*



Lemma 3.7. *Let tr_n^β be a bubble trace and x be a knot. Then the following is a Markov trace:*

$$Tr_n(x) = (\sqrt{q})^{w(x)+n} tr_n^{\frac{q+1}{q}}(\phi(x))$$

where $w(x)$ is the writhe of x.

Proof. Suppose tr_n^β is a bubble trace, with $\beta = \frac{q+1}{q}$, and consider the above definition of Tr_n. From Lemma 3.5, we see that Tr_n preserves the first Markov move. Now consider $tr_{n+1}^\beta(\phi(x\sigma_n))$, with $x \in B_{n+1}$ and x does not contain σ_n. Now define $\phi'(x)$ to be the linear combination of all the diagrams of $\phi(x)$ without 2-vertical lines in the n^{th} position. Hence, we can define

$$\begin{aligned} tr_{n+1}^\beta(\phi(x\sigma_n)) =& (t_1 + t_2 + t_3\beta + t_4 + t_5 + t_6 + t_7 + \\ & t_8 + t_9 + t_{10} + t_{11} + t_{12}\beta) tr_{n+1}^\beta(\phi'(x)) + \\ & (t_{13} + t_{14} + t_{15}\beta) tr_{n+1}^\beta(\phi(x)) \end{aligned}$$

By our choice of β and simplifying, we are left with

$$tr_{n+1}^{\frac{q+1}{q}}(\phi(x\sigma_n)) = \frac{1}{q} tr_n^{\frac{q+1}{q}}(\phi(x))$$

Now multiplying both sides of the equation by $(\sqrt{q})^{w(x\sigma_n)+n+1}$ and using the definition of Tr_n and $w(x)$, we have that

$$\begin{aligned} Tr_{n+1}(x\sigma_n) &= (\sqrt{q})^{w(x)+1+n+1-2} tr_n^{\frac{q+1}{q}}(\phi(x)) \\ &= (\sqrt{q})^{w(x)+n} tr_n^{\frac{q+1}{q}}(\phi(x)) \\ &= Tr_n(x) \end{aligned}$$

and we see the second Markov move is preserved. The third and final Markov move is proved to be preserved in a symmetric argument, and we are done. □

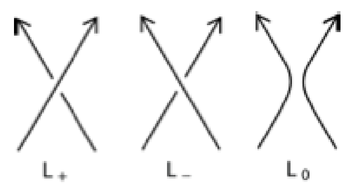
3.3. Recovering the Jones Polynomial.

Before we reach the main theorem of this chapter, we define the Jones polynomial as follows:

Definition 3.8. *The Jones polynomial is a knot polynomial V that satisfies the skein relation:*

$$t^{-1}V_{L_+} - tV_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}$$

where t is a scalar, and L_+, L_- and L_0 are defined below:



Now we have all of the tools to prove the following theorem.

Theorem 3.9. *If $q \neq -1, 0$, then $Tr_n(x) = \frac{q+1}{\sqrt{q}} V(\hat{x})$, where $x \in B_n$, $\hat{x}$ is the braid closure of x, and V is the Jones Polynomial.*

Proof. Consider Tr_n and recall that it is a Markov trace by Lemma 3.6. Since it is a Markov trace, it is an invariant of the oriented knot, and, moreover, if it satisfies the Jones skein relation, it must be a scalar multiple of the Jones polynomial. From the above definition, Lemma 3.2, and applying $tr_n^{\frac{q+1}{q}}$ to both sides, we see that

$$\begin{aligned} \phi(x\sigma_i) - q\phi(x\sigma_i^{-1}) &= (1-q)\phi(x) \\ \Rightarrow tr_n^{\frac{q+1}{q}}(\phi(x\sigma_i)) - q tr_n^{\frac{q+1}{q}}(\phi(x\sigma_i^{-1})) &= (1-q) tr_n^{\frac{q+1}{q}}(\phi(x)) \end{aligned}$$

Substituting in the definition of Tr_n, we have

$$\begin{aligned} tr_n^{\frac{q+1}{q}}(\phi(x\sigma_i)) - q tr_n^{\frac{q+1}{q}}(\phi(x\sigma_i^{-1})) &= (1-q) tr_n^{\frac{q+1}{q}}(\phi(x)) \\ \Rightarrow (\sqrt{q})^{-(w(x)+n+1)} Tr_n(\phi(x\sigma_i)) - q(\sqrt{q})^{-(w(x)+n+1)} Tr_n(\phi(x\sigma_i^{-1})) &= (1-q)(\sqrt{q})^{-(w(x)+n+1)} Tr_n(\phi(x)) \\ \Rightarrow \frac{1}{q} Tr_n(\phi(x\sigma_i)) - q Tr_n(\phi(x\sigma_i^{-1})) &= (\frac{1}{\sqrt{q}} - \sqrt{q}) Tr_n(\phi(x)) \end{aligned}$$

and we see that this is precisely the Jones skein relation. Now, to find the aforementioned scaler of the Jones polynomial, we must see what our Markov trace sends the unknot to:

$$\begin{aligned} Tr_1(\hat{\mathbb{1}}) &= \sqrt{q}^{0+1} tr_1^{\frac{q+1}{q}} \\ &= \sqrt{q}\frac{q+1}{q} \\ &= \frac{q+1}{\sqrt{q}} \end{aligned}$$

Therefore, $Tr_n(x) = \frac{q+1}{\sqrt{q}} V(\hat{x})$. $\square$

4. The c-color Generalization

4.1. **Background Notation.**

In the previous results, we worked with a 2-colored Planar Rook Algebra, and now we wish to extend this to c-colors. First off, we extend our definition to satisfy our generalization

Definition 4.1. *$P_{n,c}$ is a monoid whose elements are all the possible diagram constructed in the following way:*

1. *We first start with a rectangle with n nodes on top, and n nodes on bottom.*
2. *We connect nodes by lines, known as "edges." Each edge connects a node at the top to a node at the bottom. Each node is an endpoint of either one or zero edges.*
3. *Each edge has one of c colors and we require that no two edges of the same color can intersect when drawn. Further, we establish a hierarchy on colors through labeling the colors from 1 to c.*

Given $a, b \in P_{n,c}$, our monoid operation is defined by an algorithm:

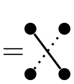
1. *Identify the bottom row of vertices in a with the top row of vertices in b. (We now have a diagram with three rows of vertices).*
2. *Delete all edges that terminate at the middle row (i.e. connect do not connect to another edge).*
3. *Think of two connecting edges as one long edge and delete the middle row of vertices.*
4. *If we have an edge with color label c_1 and an edge with color label c_2, where $c_1 \neq c_2$, then we delete the edges.*

Before we can define and prove that there is a homomorphism between B_n and $\mathbb{C}P_{n,c}$, we establish a set T and an action of $P_{2,c}$ on T.

Definition 4.2.

$$T = \{(c_l, c_m) | c_l, c_m \in \{0, 1, \ldots, c\}\}$$

Definition 4.3. *Given a diagram in $d \in P_{2,c}$, we define a 4-tuple $\delta = (c_1, c_2, c_3, c_4)$ to be the the colors of the nodes of d, where c_1 corresponds to the top left color, c_2 corresponds to the top right color, c_3 corresponds to the bottom left color, and c_4 corresponds to the bottom right color. For example, in the two color case, if*

$d =$ 

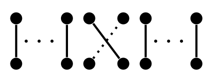
Then, $\delta = (1, 2, 2, 1)$.

Definition 4.4. *We define an action of $d \in P_{2,c}$ on $(c_l, c_m) \in T$ by first taking the $\delta = (c_1, c_2, c_3, c_4)$ corresponding to d, and then defining:*

$$d \cdot (c_l, c_m) = \begin{cases} 0, & c_l \neq c_1 \text{ or } c_m \neq c_2 \\ (c_3, c_4), & c_l = c_1 \text{ and } c_m = c_2 \end{cases}$$

Next, we extend the definition of δ so that we can define the action of $\mathbb{C}P_{2,c}$ on T.

Definition 4.5. *Given a diagram in $d_{ki} \in \mathbb{C}P_{n,c}$, we define a 4-tuple $\delta_{ki} = (c_1, c_2, c_3, c_4)$ to be the the colors of the nodes of d_{ki} restricted the the i^{th} and $i+1^{st}$ nodes on top and bottom, where c_1 corresponds to the top left color, c_2 corresponds to the top right color, c_3 corresponds to the bottom left color, and c_4 corresponds to the bottom right color. For example, in the two color case, if*

$d_{ki} =$ 

Then, $\delta_{ki} = (1, 2, 2, 1)$.

Definition 4.6. *We define an action of $d_{ki} \in \mathbb{C}P_{n,c}$ on $(c_l, c_m) \in T$ by first taking the $\delta_{ki} = (c_1, c_2, c_3, c_4)$ corresponding to d_{ki}, and then defining:*

$$d_{ki} \cdot (c_l, c_m) = \begin{cases} 0, & c_l \neq c_1 \text{ or } c_m \neq c_2 \\ (c_3, c_4), & c_l = c_1 \text{ and } c_m = c_2 \end{cases}$$

Finally, if we extend T to be tuples of length n,

Definition 4.7. *We define an action of $d_{ki} \in \mathbb{C}P_{n,c}$ on $(c_{l_1}, \ldots, c_{l_i}, c_{l_{i+1}}, \ldots, c_{l_n}) \in T$, and by first taking the $\delta_{ki} = (c_1, c_2, c_3, c_4)$ corresponding to d_{ki},*

$$d_{ki} \cdot (c_{l_1}, \ldots, c_{l_n}) = \begin{cases} 0, & c_{l_i} \neq c_1 \text{ or } c_{l_{i+1}} \neq c_2 \\ (c_3, c_4), & c_{l_i} = c_1 \text{ and } c_{l_{i+1}} = c_2 \end{cases}$$

With these definitions in hand, we can now define the c-color homomorphism.

4.2. **The c-color Homomorphism.**

Theorem 4.8. *The map* $\phi_{n,c} : B_n \to \mathbb{C}P_{n,c}$ *given by* $\phi_{n,c}(\sigma_i) = \sum_{k=1}^{|P_{n,c}|} t_k d_{k,i}$ *is a homomorphism for the following* t_k*'s:*

$$t_{i,j,i,j} = \begin{cases} 0, & i > j > 0 \\ 1-q, & j > i > 0 \end{cases}$$

$$t_{i,j,j,i} = \begin{cases} 1, & i > j > 0 \\ q, & j > i > 0 \end{cases}$$

$$t_{i,i,i,i} = \begin{cases} 1, & i > 0 \\ 1 - \left(\sum_{\{a_1,a_2,a_3,a_4\} \neq \{0,0,0,0\}} t_{a_1,a_2,a_3,a_4} \right), & i = 0 \end{cases}$$

$$t_{0,j,0,j} = (2-j)(1-q) - 1$$
$$t_{j,0,j,0} = -(c-j) \cdot (1-q) - 1$$
$$t_{0,j,j,0} = q(2-j) - (c-j)$$
$$t_{i,0,0,i} = (2-j) - q(c-j).$$

Lemma 4.9. $\phi_{n,c}$ *satisfies the Braid relations if the the action of* $\phi_{n,c}$ *on* T *satisfies the following relations on* $\mathbb{C}T$*:*

$$\phi_{n,c}(\sigma_i) \cdot (i,j) = \begin{cases} (j,i), & i > j \\ (1-q)(i,j) + q(j,i), & i < j \\ 1, & i = j \end{cases} .$$

Proof of Lemma 4.9. Assume $\phi_{n,c}(\sigma_i)$ satisfies the relation above. To show $\phi(\sigma_{i+1})\phi(\sigma_i)\phi(\sigma_{i+1}) = \phi(\sigma_i)\phi(\sigma_{i+1})\phi(\sigma_i)$, it suffices to check that for arbitrary colored tuple, $(*, ..., r, s, t, ..., *)$, where r occurs in the i^{th} position, the action $\phi(\sigma_i)\phi(\sigma_{i+1})\phi(\sigma_i)$ on $(*, ..., r, s, t, ..., *)$ yields the same result as the action of $\phi(\sigma_{i+1})\phi(\sigma_i)\phi(\sigma_{i+1})$ on $(*, ..., r, s, t, ..., *)$. Since the actions of d_{ki} and $d_{k(i+1)}$ on $(*, ..., r, s, t, ..., *)$ only affect the i'th, $(i+1)$'th and $(i+2)$'th positions, for notational purposes, write $(r, s, t) = (*, ..., r, s, t, ..., *)$.

Now, $r, s, t \in \{0, \cdots, c\}$ and the assumed relation for $\phi_{n,c}(\sigma_i)$ depends on the relative sizes of r, s, and t. So, each of the following cases must be considered separately:

(1) $r = s, s < t$
(2) $r = s, s > t$
(3) $r = t, s < t$
(4) $r = t, s > t$
(5) $r < s < t$
(6) $r > s > t$
(7) $s > r > t$
(8) $s > t > r$
(9) $s < r < t$
(10) $s < t < r$

First, consider the case $r = s$ and $s < t$.

$$\begin{aligned}
(\phi(\sigma_i)\phi(\sigma_{i+1})\phi(\sigma_i)) \cdot (r,s,t) &= (\phi(\sigma_i)\phi(\sigma_{i+1})) \cdot (\phi(\sigma_i) \cdot (r,s,t)) \\
&= (\phi(\sigma_i)\phi(\sigma_{i+1})) \cdot (r,s,t), \text{ Since } r = s \text{ and } \phi(\sigma_i) \text{ does not act on } t. \\
&= \phi(\sigma_i) \cdot (\phi(\sigma_{i+1}) \cdot (r,s,t)) \\
&= \phi(\sigma_i) \cdot ((1-q)(r,s,t) + q(r,t,s)), \ \phi(\sigma_{i+1}) \text{ does not act on } r \text{ and } s < t \\
&= (1-q)\phi(\sigma_i) \cdot (r,s,t) + q\phi(\sigma_i) \cdot (r,t,s) \\
&= (1-q)(r,s,t) + q[(1-q)(r,t,s) + q(t,r,s)] \\
&= (1-q)(r,s,t) + q(1-q)(r,t,s) + q^2(t,r,s). \quad \star \\
(\phi(\sigma_{i+1})\phi(\sigma_i)\phi(\sigma_{i+1})) \cdot (r,s,t) &= (\phi(\sigma_{i+1})\phi(\sigma_i)) \cdot (\phi(\sigma_{i+1}) \cdot (r,s,t)) \\
&= (\phi(\sigma_{i+1})\phi(\sigma_i)) \cdot ((1-q)(r,s,t) + q(r,t,s)) \\
&= \phi(\sigma_{i+1}) \cdot (\phi(\sigma_i) \cdot ((1-q)(r,s,t) + q(r,t,s)) \\
&= \phi(\sigma_{i+1}) \cdot ((1-q)(r,s,t) + q[(1-q)(r,t,s) + q(t,r,s)]) \\
&= (1-q)[(1-q)(r,s,t) + q(r,t,s)] + q[(1-q)(r,s,t) + q(t,r,s)] \\
&= (1-q)(r,s,t) + q(1-q)(r,t,s) + q^2(t,r,s). \quad \star
\end{aligned}$$

Comparing the stared equations shows that when $r = t$ and $s < t$, the action of $\phi(\sigma_i)\phi(\sigma_{i+1})\phi(\sigma_i)$ on $(*, ..., r, s, t, ..., *)$ yields the same result as the action of $\phi(\sigma_{i+1})\phi(\sigma_i)\phi(\sigma_{i+1})$ on $(*, ..., r, s, t, ..., *)$. Similar calculations show the remaining 9 cases hold.

□

Proof of Theorem 4.8. The only counting comes into play when calculating the coefficients of $t_{0,i,0,i},\ t_{i,0,i,0},\ t_{0,i,i,0},\ t_{i,0,0,i}$. Notice $\phi_{n,c}(\sigma_i)(0,j) = (1-q)(0,j) + q(j,0)$ and every diagram with a four digit vertex sequence of the form (i,j,i,j) will map $(0,j)$ to itself. So the element in the Burau Block corresponding to $(0,j),(0,j)$ is the sum

$$\sum_{i \neq j} t_{i,j,i,j} = 1 - q.$$

We know this sum equals $1-q$ because $\phi_{n,c}(\sigma_i)(0,j) = (1-q)(0,j) + q(j,0)$. We can then pull out the term of interest and rearrange the equality to yield

$$\begin{aligned}
t_{0,j,0,j} &= 1 - q - \sum_{0 \neq i \neq j} (t_{i,j,i,j}) \\
&= 1 - q - \sum_{i=1}^{i=j-1} (t_{i,j,i,j}) - \sum_{i=j+1}^{c} (t_{i,j,i,j}) - t_{j,j,j,j} \\
&= 1 - q - \sum_{i=1}^{i=j-1} (1-q) - \sum_{i=j+1}^{c} 0 - 1 \\
&= (1-q) - (j-1) \cdot (1-q) - 1 \\
&= (2-j)(1-q) - 1
\end{aligned}$$

$$\begin{aligned}
t_{j,0,j,0} &= -\sum_{0 \neq i \neq j} (t_{j,i,j,i}) \\
&= -\sum_{i=1}^{i=j-1} (t_{j,i,j,i}) + \sum_{i=j+1}^{c} (t_{j,i,j,i}) - t_{j,j,j,j} \\
&= -\sum_{i=1}^{i=j-1} 0 + \sum_{i=j+1}^{c} 1 - q - 1 \\
&= -(c-j) \cdot (1-q) - 1
\end{aligned}$$

$$\begin{aligned} t_{0,j,j,0} &= q - \sum_{0\neq i\neq j} (t_{i,j,j,i}) \\ &= q - \sum_{i=1}^{i=j-1} (t_{i,j,j,i}) - \sum_{i=j+1}^{c} (t_{i,j,j,i}) \\ &= q - \sum_{i=1}^{i=j-1} (q) - \sum_{i=j+1}^{c} 1 \\ &= q - (j-1)\cdot(q) - (c-j) \\ &= q(2-j) - (c-j) \end{aligned}$$

$$\begin{aligned} t_{i,0,0,i} &= 1 - \sum_{0\neq j\neq i} (t_{i,j,j,i}) \\ &= 1 - \sum_{j=1}^{j=i-1} (t_{i,j,j,i}) - \sum_{j=i+1}^{c} (t_{i,j,j,i}) \\ &= 1 - \sum_{j=1}^{j=i-1} 1 - \sum_{j=i+1}^{c} q \\ &= 1 - (c-j)\cdot(q) - (j-1) \\ &= (2-j) - q(c-j)\ . \end{aligned}$$

Before continuing we need to address a very simple object, the identity in $\mathbb{C}P_{n,c}$. $e \in \mathbb{C}P_{1,c}$ is the linear combination of every colored edge such that every non-zero color has a coefficient of 1 and the coefficient of the zero edge is $c-1$ where, as usual, c is the number of colors being used. Just as before $e^{\otimes n}$ is the identity in $\mathbb{C}P_{n,c}$.

Thus the coefficients are completely determined for every $\phi_{n,c}(\sigma_i)$.

$\square$

For further computation, the coefficients of $\phi_{n,c}(\sigma_i^{-1})$ are needed . Using the fact that $\phi_{n,c}$ is a homomorphism, $(\phi_{n,c}(\sigma_i)-1)(\phi_{n,c}(\sigma_i)+qe^{\otimes n}) = \overrightarrow{0}$. Or, equivalently, $\phi^2_{n,c}(\sigma_i)+(q-1)(\phi_{n,c}(\sigma_i)) - qe^{\otimes 2} = \overrightarrow{0}$. Again, for this it suffices to check the action of $\phi^2_{n,c}(\sigma_i) + (q-1)(\phi_{n,c}(\sigma_i)) - qe^{\otimes n}$ on a tuple (i,j) produces the zero vector. (Notice how much nicer this is computationally than computing representation matrices and carrying out the full matrix multiplication). Again, we shall show one example to demonstrate the point and the rest follows similarly. For $i > j$,

$$\begin{aligned} &(\phi^2_{n,c}(\sigma_i) + (q-1)(\phi_{n,c}(\sigma_i)) - qe^{\otimes n})\cdot(i,j) \\ &= \phi^2_{n,c}(\sigma_i)\cdot(i,j) + (q-1)(\phi_{n,c}(\sigma_i))\cdot(i,j) - qe^{\otimes n}\cdot(i,j) \\ &= \phi_{n,c}(\sigma_i)\cdot(\phi_{n,c}(\sigma_i)\cdot(j,i)) + (q-1)(\phi_{n,c}(\sigma_i))\cdot(i,j) - qe^{\otimes n}\cdot(i,j) \\ &= \phi_{n,c}(\sigma_i)\cdot(j,i) + (q-1)(j,i) - q(i,j) \\ &= (1-q)(j,i) + q(i,j) + (q-1)(j,i) - q(i,j) \\ &= \overrightarrow{0}\ . \end{aligned}$$

So, we have $\phi^2_{n,c}(\sigma_i) + (q-1)(\phi_{n,c}(\sigma_i)) - qe^{\otimes n} = \overrightarrow{0}$ and this is nice because multiplying by $\phi^{-1}_{n,c}$ yields a way to calculate $\phi_{n,c}(\sigma_i^{-1})$ via $\phi_{n,c}(\sigma_i)$ and $\phi_{n,c}(\mathbb{1})$,

$$\begin{aligned} \phi_{n,c}(\sigma_i^{-1})(\phi^2_{n,c}(\sigma_i) + (q-1)(\phi_{n,c}(\sigma_i)) - qe^{\otimes n}) &= \phi^{-1}_{n,c}(\sigma_i)(\phi^2_{n,c}(\sigma_i) + (q-1)(\phi_{n,c}(\sigma_i)) - qe^{\otimes n}) \\ &= \phi_{n,c}(\sigma_i) + (q-1)e^{\otimes n} - q\phi^{-1}_{n,c}(\sigma_i) = \overrightarrow{0} \\ &\Rightarrow \phi^{-1}_{n,c}(\sigma_i) = \frac{\phi_{n,c}(\sigma_i) + (q-1)e^{\otimes n}}{-q} \end{aligned}$$

and we have an expression for $\phi_{n,c}(\sigma_i^{-1})$.

5. Conclusion

Conclude stuff here!

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