SPECIAL ELEMENTS OF THE HECKE AND TEMPERLEY-LIEB ALGEBRAS
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Abstract

The Hecke algebra $H_n(k)$ and Temperley-Lieb algebra $TL_n(q)$ are very neatly related, for example there is a surjective algebra homomorphism from the Hecke algebra to the basis of the Temperley-Lieb algebra. Using this, we examine various special elements of the Hecke algebra - including the Murphy operators and several idempotents - as elements of Temperley-Lieb algebra under this homomorphism. Because the Temperley-Lieb algebra can be represented by tangles of string in a three dimensional space, it is sometimes possible to create simple and elegant representations that make algebraic properties visually obvious.

1. Background

1.1. Basic definitions.

Definition 1.1. *An* algebra *is a vector space V over a field K with an operation $\cdot : V \otimes V \to V$, such that $\cdot$ is bilinear. That is, given vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, we have that $(\mathbf{x}+\mathbf{y})\cdot\mathbf{z} = \mathbf{x}\cdot\mathbf{z}+\mathbf{y}\cdot\mathbf{z}$ (left distributivity), $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = \mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z}$ (right distributivity), and, given $a, b \in K$, $(a\mathbf{x})\cdot(b\mathbf{y}) = (ab)(\mathbf{x}\cdot\mathbf{y})$ (scalar compatibility).*

Definition 1.2. *An* associative algebra *is an algebra in which multiplication is associative, and so, an associative algebra has the properties of both a ring and a vector space. In this paper, we assume all of our associative algebras are* unital, *ie, they have a multiplicative identity.*

Definition 1.3. *The* group algebra $\mathbb{C}[S_n]$ of the symmetric group *is the algebra over the field $\mathbb{C}$ where the basis vectors are permutations on n elements; ie, precisely the elements of S_n. For instance, in S_4, various elements include $5(213)$, $(e^{\cdot 5i})(12)(34)$ and $(4+7i)(3142)$. The vector multiplication operation is simply the group operation in S_n, extended in the unique bilinear way to $\mathbf{C}S_n$. For example, given $(ij), (kl), (st) \in S_n$, and $a, b \in \mathbb{C}$ we have that:*

$$\begin{aligned} (ij)\cdot[(kl)+(st)] &= (ij)(kl)+(ij)(st) \\ (ij)(st)+(kl)(st) &= [(ij)+(kl)]\cdot(st) \\ (a(ij))\cdot(b(kl)) &= (ab)((ij)(kl)) \end{aligned}$$

From this point on, given $\mathbf{x}$ and $\mathbf{y}$, we write $\mathbf{xy}$ for $\mathbf{x}\cdot\mathbf{y}$.

Additionally, we can define $\mathbb{C}[S_n]$ via generators and relations.

Proposition 1.4. $\mathbb{C}[S_n]$ *is spanned by elements generated by* $\sigma_i = (i\ i+1)$, $1 \leq i \leq n-1$, *with defining relations:*

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{if } |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i^2 &= 1 \end{aligned}$$

Proof. First, we know that in S_n, the σ_i are in fact a generating set. So, any element of S_n can be written as a product of the σ_i. Now, we need to show that the relations given hold for S_n. □

1.2. **The Hecke Algebra.**

Definition 1.5. *The* Iwahori-Hecke Algebra*, or Hecke Algebra, denoted* $H_n(s)$*, is an algebra with generators* $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$*, subject to the following relations:*

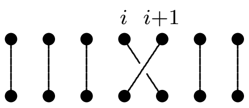
$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{if } |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i^2 &= (s-s^{-1})\sigma_i + 1 \end{aligned}$$

where s *is a fixed invertible element of the field.*

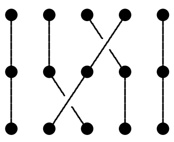
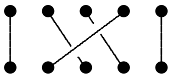
For our purposes, we can use any field such that we can find an element s such that $s - s^{-1} \neq 0$; several common fields for this include $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are common choices.

Observation 1.6. *If we let* $s = 1$*, then* $H_n(s) \cong \mathbb{C}[S_n]$

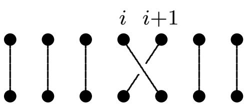
Now, we can represent the σ_i visually in the following manner:

$\sigma_i =$ 

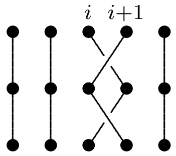
Multiplication of two diagrams is handled simply by stacking the two on top of each other; for instance, in $H_4(s)$,

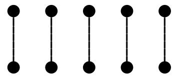
$\sigma_3\sigma_2 =$  $=$ 

From these diagrams, we have that σ_i has an inverse, σ_i^{-1} where the diagram for σ_i^{-1} is

$\sigma_i^{-1} =$ 

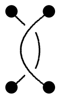
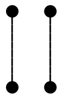
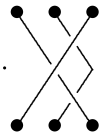
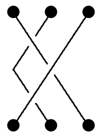
since, clearly,

$\sigma_i\sigma_i^{-1} =$ 

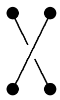
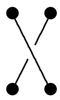
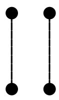
$=$  $= e.$

It show be clear that this diagram, e, is the identity.

1.3. **Skein Relations.** While we have the set of relations defined for $H_n(s)$, it is useful to be able to express these relations in diagram form. Two diagrams are equivalent if they can be transformed into each other by a series of the second and third *Reidemeister moves*, which are depicted below:

2.  $=$  3.  $=$  .

These three moves can be expressed as 2) moving one stand completely over another or 3) moving a stand over or under a crossing. Notice that, in $H_n(s)$, the second move is precisely how we evaluate inverses. In order to properly use these moves in $H_n(s)$, we need to add an additional relation:

 $-$  $= (s - s^{-1})$  , (1)

or, equivalently,

$$\sigma_i - \sigma_i^{-1} = s - s^{-1}. \tag{2}$$

This relation is known as the skein relation.

Proposition 1.7. *The above skein relation is equivalent to the quadratic relation, and Reidemeister moves 2 and 3 can be expressed by simple relations in $H_n(s)$.*

Proof. First, we show the equivalence with the quadratic relation - which is a matter of simple algebra:

$$\begin{aligned} \sigma_i - \sigma_i^{-1} &= s - s^{-1} \\ \sigma_i^2 - 1 &= (s - s^{-1})\sigma_i \\ \sigma_i^2 &= (s - s^{-1})\sigma_i + 1. \end{aligned}$$

The proof of the second Reidemeister move is simply a matter of observing that $H_n(s)$ has inverses, and we have the third Reidemeister move from the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ - the diagrams obtained by this combination of elements is precisely Reidemeister 3. □

It

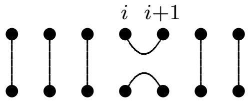
1.4. **The Temperley-Lieb Algebra.**

Definition 1.8. *The* Temperley-Lieb Algebra *over a ring, $TL_n(\delta)$ $(n \geq 1)$, is an associative algebra with generators $e_1, e_2, \ldots, e_{n-1}$ and the following relations:*

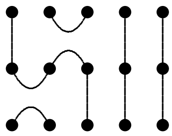
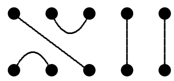
$$\begin{aligned} e_i^2 &= \delta e_i \\ e_i e_{i\pm1} e_i &= e_i \\ e_i e_j &= e_j e_i \quad |i-j| > 1 \end{aligned}$$

where δ is a fixed invertible scalar in the ring.

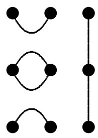
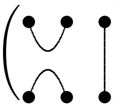
Similarly to the Hecke algebra, there is a very natural way to express elements of this algebra as diagrams.

$e_i =$ 

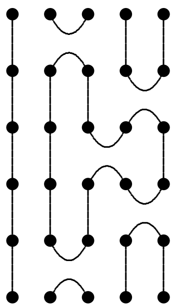
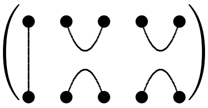
Again, similarly to the multiplication of diagrams in $H_n(s)$, we express multiplication in $TL_n(\delta)$ by stacking. For an example, in $TL_5(\delta)$:

$e_2e_1 =$  $=$ 

Sometimes, when elements are stacked, we create a closed loop in the diagrams. Whenever this occurs, we remove the loop and replace it with δ. For instance, in $TL_3(\delta)$,

$e_1^2 =$  $= \delta$ () $= \delta e_1$

which is precisely our quadratic operator. This loop-removal process also applies to larger loops as well, such as:

$e_2e_4e_3e_4e_2 =$  $= \delta$ () $= \delta(e_2e_4)$

We can show this via the relations by noticing that $e_2(e_4e_3e_4)e_2 = e_2(e_4)e_2 = e_2e_2e_4 = \delta e_2e_4$

Notice that any identity element of a function is an idempotent.

1.5. **The Jucys-Murphy Elements.**

Definition 1.9. *The* Jucys-Murphy Elements *in* $\mathbb{C}[S_n]$ *are defined as the following sum:*

$$m(j) = \sum_{i=1}^{j-1} (ij) \in \mathbb{C}[S_n]$$

for $j = 2, \dots, n$.

Proposition 1.10. *Given* $m(i), m(j) \in \mathbb{C}[S_n]$, $m(i)m(j) = m(j)m(i)$.

Proof. First, notice that $m(2)m(3) = (12) * [(13) + (23)] = (132) + (123) = m(3)m(2)$. Now, assume $j > k > i$. Now, we show $(ik)m(j) = m(j)(ik)$

$$\begin{aligned} (ik)m(j) &= (ik)[(1j) + (2j) + \cdots + (ij) + \cdots + (kj) + \cdots + (j-1j)] \\ &= (ik)(1j) + (ik)(2j) + \cdots + (ik)(ij) + \dots (ik)(kj) + \dots (ik)(j-1j) \\ &= (1j)(ik) + (2j)(ik) + \cdots + (ijk) + \cdots + (ikj) + \dots (j-1j)(ik) \\ &= (1j)(ik) + (2j)(ik) + \cdots + (ij)(ik) + \cdots + (kj)(ik) + \dots (j-1j)(ik) \\ &= m(j)(ik) \end{aligned}$$

Now, (assuming, without loss of generality, that $j > i$)

$$\begin{aligned} m(i)m(j) &= \sum_{k=1}^{i-1} (ki)m(j) \\ &= \sum_{k=1}^{i-1} m(j)(ki) \\ &= m(j)m(i) \end{aligned}$$

□

Proposition 1.11. $\sum_{i=1}^{n} m(i) = M_n \in Z(\mathbb{C}[S_n])$

Proof. Proved in [1]. □

2. Construction of the Murphy Elements in $H_n(k)$

First, note that there is a clear surjective mapping $\psi : H_n(s) \to \mathbb{C}[S_n]$, $\psi(\sigma_i) = (i\ i+1)$; in other words, $\mathbf{C}[S_n]$ is a specialization of $H_n(1)$. For an element of $r \in H_n(k)$ to be an analog of a Jucys-Murphy element, $\psi(r)$ must be $m(j)$ for some j. It such be immediately clear that $\psi(\sigma_1) = (12) = m(2)$. Less clear are the construction of other Jucys-Murphy elements, but, if first we notice that

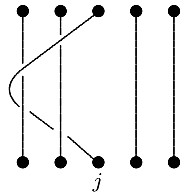
$$m(j) = \sum_{i=1}^{j-1} (ij) = \sum_{i=1}^{j-1} \big((j-1j)(j-2j)\dots(i+1j)(ij)(i+1j)\dots(j-2j)(j-1j)\big) \qquad (3)$$

The construction in $H_n(s)$ becomes readily apparent.

Definition 2.1. *In $H_n(s)$, $m(j)$ can be represented as*

$$M(j) = \sum_{i=1}^{j-1} (\sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i\sigma_{i+1}\ldots\sigma_{j-2}\sigma_{j-1}) \tag{4}$$

Additionally, if we let

$T(j) =$ 

$= \sigma_{j-1}\ldots\sigma_2\sigma_1^2\sigma_2\ldots\sigma_{j-1}$ (5)

Proposition 2.2.

$$M(j) = \frac{T(j)-1}{s-s^{-1}} \tag{6}$$

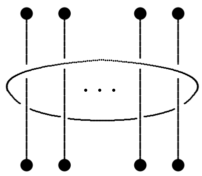
Proof. We show that (6) holds. Notice that:

$$\begin{aligned}
T(j) &= \sigma_{j-1}\ldots\sigma_2\sigma_1^2\sigma_2\ldots\sigma_{j-1} \\
&= \sigma_{j-1}\ldots\sigma_2((s-s^{-1})\sigma_1+1)\sigma_2\ldots\sigma_{j-1} \\
&= (s-s^{-1})\sigma_{j-1}\ldots\sigma_2\sigma_1\sigma_2\ldots\sigma_{j-1} + \sigma_{j-1}\ldots\sigma_2^2\ldots\sigma_{j-1} \\
&= (s-s^{-1})\left(\sigma_{j-1}\ldots\sigma_2\sigma_1\sigma_2\ldots\sigma_{j-1} + \sigma_{j-1}\ldots\sigma_2\ldots\sigma_{j-1}\right) \\
&+ \sigma_{j-1}\ldots\sigma_3^2\ldots\sigma_{j-1} \\
&= (s-s^{-1})\left(\sum_{i=1}^{j-1}\left(\sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i\sigma_{i+1}\ldots\sigma_{j-2}\sigma_{j-1}\right)\right)
\end{aligned}$$

□

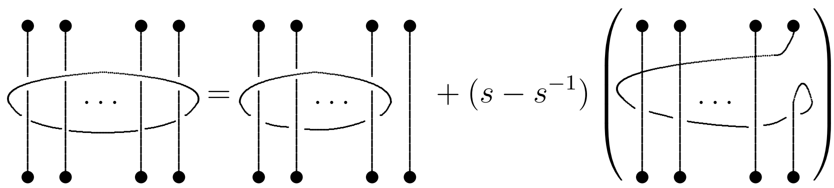
Additionally, we now consider $M = \sum_{j=1}^{n} M(j)$.

Definition 2.3. *We can represent M as a linear combination of $T^{(n)}$ and the identity,*

$T^{(n)} =$ 

and $T^{(0)} = e$.

Proof. First, we decompose one of the crossings of $T^{(n)}$ Notice that:



Or, alternatively,

$$T^{(n)} = T^{(n-1)} + (s-s^{-1})T(n)$$

since the loop in the last diagram can be resolved by Reidemeister 1. Continuing this pattern, we get that

$$T^{(n)} = (s-s^{-1})\sum_{i=1}^{n} T(i) + T^{(0)}$$

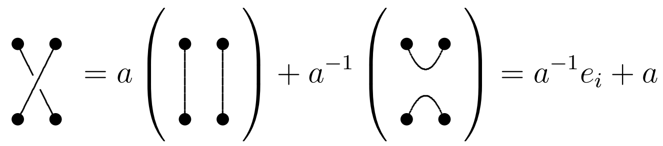
and since we have that $T(j) = (s - s^{-1})M(i) + 1$, we ge that

$$\begin{aligned} T^{(n)} &= (s - s^{-1}) \sum_{i=1}^{n} \big((s - s^{-1})M(i) + 1\big) + 1 \\ &= (s - s^{-1})^2 \sum_{i=1}^{n} M(i) + n \\ M &= \frac{T^{(n)} - n}{(s - s^{-1})^2} \end{aligned}$$

□

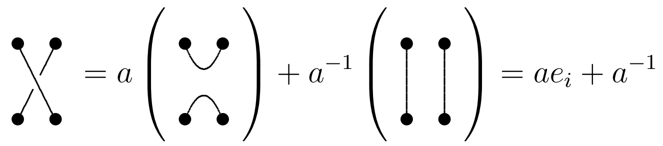
3. Imposing the Skein Relations on $TL_n(\delta)$

In order to be able to better view the Murphy elements in $TL_n(\delta)$, we need to define the notion of crossings in the algebra. So, we follow Kauffman, and define



(7)

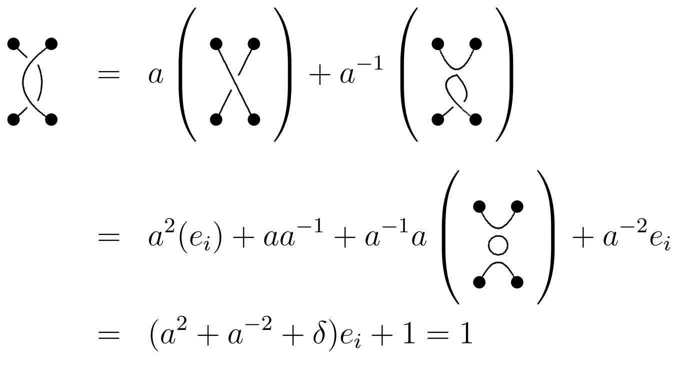
Or, for the opposite facing crossing,



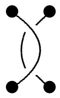
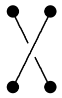
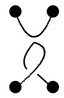
(8)

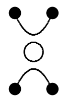
where $a^2 + a^{-2} = -\delta$.

We now need to show that these relations allow Reidemeister moves 2 and 3 to hold. Given these relations, notice that:



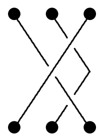
Notice that the other side of Reidemeister 2,

 $= a^{-1}\Big($  $\Big) + a\Big($  $\Big)$

$= a^{-1}(a^{-1}e_i + a) + a\Big(a^{-1}$  $+ ae_i\Big)$

$= (a^2 + a^{-2} + \delta)e_i + 1 = 1$

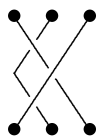
Now, we show that Reidemeister 3 holds as well. Notice that:

 $= (ae_2 + a^{-1})(a^{-1}e_1 + a)(a^{-1}e_2 + a).$

by converting from the diagram into the algebraic expression. Now,

$$\begin{aligned}
& (ae_2 + a^{-1})(a^{-1}e_1 + a)(a^{-1}e_2 + a) \\
= \; & (e_2e_1 + a^{-2}e_1 + a^2e_2 + 1)(a^{-1}e_2 + a) \\
= \; & a^{-1}e_2 + a^{-3}e_1e_2 + ae_2^2 + a^{-1}e_2 + ae_2e_1 + a^{-1}e_1 + a^3e_2 + a \\
= \; & (2a^{-1} + a\delta + a^3)e_2 + a^{-1}e_1 + a^{-3}e_1e_2 + ae_2e_1 + a \\
= \; & (2a^{-1} + a(-a^2 - a^{-2}) + a^3)e_2 + a^{-1}e_1 + a^{-3}e_1e_2 + ae_2e_1 + a \\
= \; & a^{-1}e_2 + a^{-1}e_1 + a^{-3}e_1e_2 + ae_2e_1 + a.
\end{aligned}$$

Similarly, the other diagram can be expressed as:

 $= (a^{-1}e_1 + a)(a^{-1}e_2 + a)(ae_1 + a^{-1}).$

which can be written as:

$$a^{-1}e_1 + ae_2e_1 + a^{-3}e_1e_2 + a^{-1}e_2 + a$$

Notice that these two expressions are identical. Additionally, at this point, for the sake of brevity, we will write ρ_i for $a^{-1}e_i + a$, and ρ_i^{-1} for $ae_i + a^{-1}$.

4. The Homomorphism $\phi : H_n(s) \to TL_n(\delta)$

Proposition 4.1. *There exists an surjective algebra homomorphism $\phi : H_n(s) \to TL_n(\delta)$ such that $\phi(\sigma_i) = e_i - s^{-1}$, and $\delta = s + s^{-1}$. Additionally, this homomorphism has a kernel generated by $K = s^{-3} + s^{-2}\sigma_1 + s^{-2}\sigma_2 + s^{-1}\sigma_1\sigma_2 + s^{-1}\sigma_2\sigma_1 + \sigma_2\sigma_1\sigma_2$.*

Proof. First, we show that this is, in fact, a homomorphism:

Given $r\sigma_i$, then $\phi(r\sigma_i) = re_i + rs^{-1} = r(e_i + s^{-1}) = r\phi(\sigma_i)$. Now, take $\phi(\sigma_i + \sigma_j) = e_i + s^{-1} + e_j + s^{-1} = \phi(\sigma_i) + \phi(\sigma_j)$ Finally, since that $\phi(\sigma_i\sigma_j) = (e_i + s^{-1})(e_j + s^{-1}) = \phi(\sigma_i)\phi(\sigma_j)$, we have that this is a homomorphism

Now, we show that $K \in \ker(\phi)$:

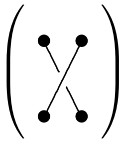
$$\begin{aligned}
\phi(K) &= \phi(s^{-3} + s^{-2}\sigma_1 + s^{-2}\sigma_2 + s^{-1}\sigma_1\sigma_2 + s^{-1}\sigma_2\sigma_1 + \sigma_2\sigma_1\sigma_2) \\
&= s^{-3} + s^{-2}(e_1 - s^{-1}) + s^{-2}(e_2 - s^{-1}) + s^{-1}(e_1 - s^{-1})(e_2 - s^{-1}) + s^{-1}(e_2 - s^{-1})(e_1 - s^{-1}) \\
&\quad + (e_2 - s^{-1})(e_1 - s^{-1})(e_2 - s^{-1}) \\
&= s^{-3} + s^{-2}e_1 - s^{-3} + s^{-2}e_2 - s^{-3} + (s^{-1}e_1 - s^{-2})(e_2 - s^{-1}) + (s^{-1}e_2 - s^{-2})(e_1 - s^{-1}) \\
&\quad + (e_2 - s^{-1})(e_1 - s^{-1})(e_2 - s^{-1}) \\
&= s^{-2}e_1 + s^{-2}e_2 - s^{-3} + s^{-1}e_1e_2 - s^{-2}e_1 - s^{-2}e_2 + s^{-3} + s^{-1}e_2e_1 - s^{-2}e_1 - s^{-2}e_2 + s^{-3} \\
&\quad + (e_2e_1 - s^{-1}e_1 - s^{-1}e_2 + s^{-2})(e_2 - s^{-1}) \\
&= s^{-3} - s^{-2}e_1 - s^{-2}e_2 + s^{-1}e_1e_2 + s^{-1}e_2e_1 + e_2e_1e_2 - s^{-1}e_1e_2 - s^{-1}e_2^2 + s^{-2}e_2 \\
&\quad - s^{-1}e_2e_1 + s^{-2}e_1 + s^{-2}e_2 - s^{-3} \\
&= e_2e_1e_2 - s^{-1}e_2^2 + s^{-2}e_2 \\
&= e_2 - s^{-1}(\delta)e_2 + s^{-2}e_2 \\
&= e_2 - s^{-1}(s + s^{-1})e_2 + s^{-2}e_2 = 0.
\end{aligned}$$

That this generates the kernel is shown in [2].

Now, we show that this is, in fact, surjective. Notice that $\phi(\sigma_i + s^{-1}) = (e_i - s^{-1}) + s^{-1} = e_i$. Since we can represent every element in $TL_n(\delta)$ as a linear combination of products of generators, we can, for any given element x in $TL_n(\delta)$, construct an element in $H_n(s)$ such that it maps to x, and this is precisely the definition of surjectivity. □

Now, notice that since $s + s^{-1} = \delta = -a^2 - a^{-2}$, we can let $s^{-1} = -a^2$ (Alternatively, we could have set $s^{-1} = a^{-2}$, but our chosen equivalence provides more convenience). This implies that

$$\phi(\sigma_i) = e_i - s^{-1} = e_i + a^2 = a(a^{-1}e + a) = a \qquad (9)$$



The diagram in the last equation is, unshockingly, identical to a multiple of the diagram representation of σ_i in $H_n(s)$.

5. The Jucys-Murphy Elements in $TL_n(\delta)$

We know that the Jucys-Murphy Elements are of the form expressed in (4). Now, the obvious way to view these elements is under the homomorphism ϕ that we just defined. And so, we have that

$$\phi(M(j)) = \phi\left(\sum_{i=1}^{j-1}\prod_{i}^{j-1}\sigma_i\right) = \sum_{i=1}^{j-1} a^i \prod_{i}^{j-1}(a^{-1}e_i + a)$$

and that the tangle $T(j)$ can be expressed as:

$$\phi(T(j)) = a^{2j-2}\rho_{j-1}\dots\rho_2\rho_1^2\rho_2\dots\rho_{j-1}.$$

So, using these, we can reconstruct the property from (6)

Lemma 5.1.

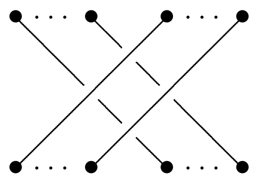
$$\phi(M(j)) = \frac{\phi(T(j)) - 1}{a^{-2} - a^2}.$$

Proof. This follows from the fact that ϕ is a homomorphism. $\square$

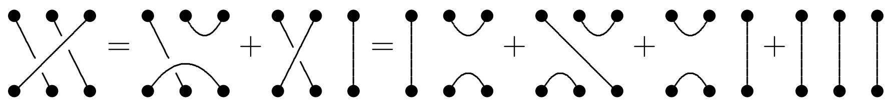
A similar result holds for $\phi(M) = \frac{\phi(T^{(n)}-n)}{(a^{-2}-a^2)^2}$.

6. Decomposition of Multiple Crossings in $TL_n(\delta)$

Now, we consider tangles in $TL_n(\delta)$ of the following form:

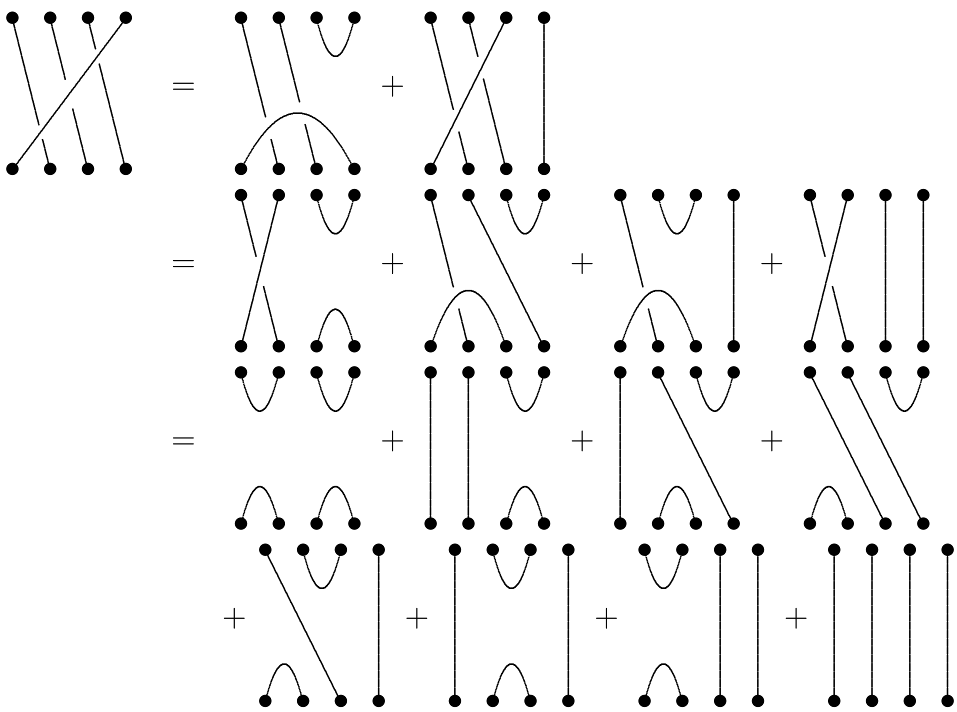


where we have n strands passing over m strands. We begin by examining the case of a single strand crossing two strands; in diagrams:



Unshockingly, this can be written as $(a^{-1}e_2 + a)(a^{-1}e_1 + a) = a^{-2}e_2e_1 + e_1 + e_2 + a^2$.

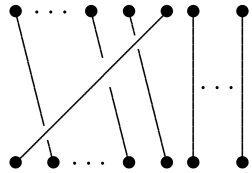
Now, we consider the diagram of a single strand crossing over 3 strands:



Similarly, it should be clear that this is $(a^{-1}e_3 + a)(a^{-1}e_2 + a)(a^{-1}e_1 + a) = a^{-3}e_3e_2e_1 + a^{-1}e_3e_2 + a^{-1}e_2e_1 + a^{-1}e_3e_1 + ae_3 + ae_2 + ae_1 + a^3$.

From these examples, the general case should be readily apparent.

Proposition 6.1. *The diagram*



a single strand crossing over n *strands in* $TL_m(\delta)$, $m > n$, *can be expressed as* $(a^{-1}e_n + a)(a^{-1}e_{n-1} + a)\dots(a^{-1}e_1 + a)$, *with a corresponding diagram decomposition.*

For the sake of brevity, we let $\xi_i = (a^{-1}e_i + a)$; we can write the above diagram as $\xi_n\xi_{n-1}\dots\xi_1$. Additionally, we denote the opposite-facing crossing as $\xi_i^{-1} = (ae_i + a^{-1})$.

Notice that:

$$\begin{aligned}\xi_i(\xi_i^{-1}) &= (a^{-1}e_i + a)(ae_i + a^{-1})\\ &= e_i^2 + a^2e_i + a^{-2}e_i + 1\\ &= (-a^2 - a^{-2})e_i + (a^2 + a^{-2})e_i + 1 = 1\end{aligned}$$

From this, we simply redefine the homomorphism $\phi : H_n(s) \to TL_n(\delta)$ as $\phi(\rho_i) = \xi_i$

7. Quantum Integers

Definition 7.1. *The* Quantum Integers *are defined as a countably infinite set of polynomials over a fixed invertible constant* q. *Let* $[n]$ *denote the nth quantum integers; then*

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \sum_{i=0}^{n} q^{n-1-2i}$$

For example, $[3] = q^2 + 1 + q^{-2}$, $[1] = \frac{q^1 - q^{-1}}{q - q^{-1}} = 1$ and $[0] = \frac{q^0 - q^{-0}}{q - q^{-1}} = 0$. It should also be clear from the definition that $[-n] = -[n]$. However, it does not follow that $[n] + [m] = [n + m]$, nor does it follow that $[n][m] = [nm]$; For example, $[2][3] = (q^1 + q^{-1})(q^2 + 1 + q^{-2}) = q^3 + 2q^1 + 2q^{-1} + q^{-3} \neq [6]$. However, there is a well-defined multiplication operation:

Proposition 7.2.

$$[n][m] = \sum_{i=1}^{n}[m + (n + 1) - 2i]$$

Proof. This follows immediately from the definition. □

From this, we get that $[2][3] = [4] + [2]$.

Definition 8.1. *In $TL_n(\delta)$, the Jones-Wenzl idempotent, denoted $f^{(n)}$, is characterised by the following properties:*

$$f^{(n)} \neq 0 \tag{10}$$

$$f^{(n)}f^{(n)} = f^{(n)} \tag{11}$$

$$e_i f^{(n)} = f^{(n)} e_i = 0 \quad \forall i \in \{1, \dots, n-1\}. \tag{12}$$

For instance, $f^{(3)} = 1 + \frac{[2]}{[3]}(e_1 + e_2) + \frac{1}{[3]}(e_1e_2 + e_2e_1)$, where $\delta = -[2]$. Notice that:

$$\begin{aligned} e_1 f^{(3)} &= e_1(1 + \frac{[2]}{[3]}(e_1 + e_2) + \frac{1}{[3]}(e_1e_2 + e_2e_1)) \\ &= e_1 + \frac{[2]}{[3]}(\delta e_1 + e_1e_2) + \frac{1}{[3]}(\delta e_1 e_2 + e_1) \\ &= \left(\frac{-[2]^2 + 1}{[3]} + 1\right) e_1 + \left(\frac{[2]}{[3]} + \frac{-[2]}{[3]}\right)(e_1e_2) \\ &= \left(\frac{-[3]}{[3]} + 1\right) e_1 = 0 \end{aligned}$$

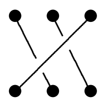
and that

$$\begin{aligned} f^{(3)} e_1 &= (1 + \frac{[2]}{[3]}(e_1 + e_2) + \frac{1}{[3]}(e_1e_2 + e_2e_1))e_1 \\ &= e_1 + \frac{[2]}{[3]}(\delta e_1 + e_2e_1) + \frac{1}{[3]}(e_1 + \delta e_2 e_1) \\ &= 0 \text{ by similar computations.} \end{aligned}$$

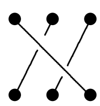
It follows identically that $f^{(3)}e_2 = e_2 f^{(3)} = 0$. Now, notice that since $\delta = -a^2 - a^{-2}$, and $[2] = q + q^{-1}$ for some q, we can let $[2] = a^2 + a^{-2} \Rightarrow q = a^2$. From this, we get that:

$$\begin{aligned} f^{(3)} &= 1 + \frac{[2]}{[3]}(e_1 + e_2) + \frac{1}{[3]}(e_1e_2 + e_2e_1) \\ ([3])f^{(3)} &= [3] + [2](e_1 + e_2) + e_1e_2 + e_2e_1 \\ &= (q^2 + 1 + q^{-2}) + (q + q^{-1})(e_1 + e_2) + e_1e_2 + e_2e_1 \\ &= (a^4 + 1 + a^{-4}) + (a^2 + a^{-2})(e_1 + e_2) + e_1e_2 + e_2e_1 \end{aligned}$$

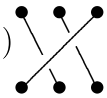
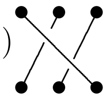
Now, by section 6, we know that: $(a^{-1}e_2 + a)(a^{-1}e_1 + a) = a^{-2}e_2e_1 + e_1 + e_2 + a^2$; and, in diagrams:


$= a^{-2}e_2e_1 + e_1 + e_2 + a^2$

and


$= a^2e_1e_2 + e_1 + e_2 + a^{-2},$

So, it follows that:

(a^2)  $+ (a^{-2})$  $+ 1 \quad = \quad ([3])f^{(3)}$

Clearly, we can view this in $H_n(s)$ by taking the inverse of ϕ (ϕ has a well-defined inverse from the image of ϕ into $H_n(s)$).

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[3] H. Morton, *Skein theory and the Murphy operators*
[4] A. Ram *Semi-normal representations of Weyl groups and Iwahori-Hecke algebras*