Solving Wave Equation Using Asymptotic Approaches

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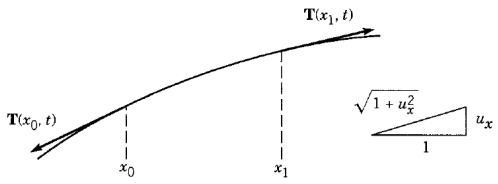
Abstract

Unlike ordinary differential equations, partial differential equations have more than one independent variables, and we need special techniques to solve these equations. We will focus on wave equations with high frequency. One of the application is that we can use such wave equations to model the primary waves of the seismic waves. The main difficulty to solve this type of equations is that for higher dimensional wave equations, it is very hard to find an analytic solution. If we try to use numerical methods, however, because of the high frequency, we will need to use many points in our calculation to get an accurate result. In order to overcome this difficulty, we will use the high frequency to separate the wave equations into many basic partial differential equations that can be easily solved. We will first study Wentzel-Kramers-Brillouin, Gaussian Beam, and Frozen Gaussian Beam. These previously studied methods provide us insights on the challenges that we will encounter when we try to solve this type of wave equations.

1 Background

1.1 Wave Equation

The main partial differential equation in this paper will be the wave equation. In order to find the formula of the equation, the following physical example can be used. Consider a flexible, elastic homogenous string of length l. Let $u(t, x)$ to be its displacement from the equilibrium position at time t and position x. Since the string is flexible, by hook's rule, the vibration of string is completely depend on the density of the string and the tension force of acting on the string. Let ρ be the density and $T(t, x)$ vector to denote the tension. We also assumed that the wave motion is purely transverse. Then we will have the following figure:



The slope of the string is denoted by $u_x(t, x)$. The assumption is that the string only has transversal movement. Thus we have the slope of the string is one in the longitudinal direction, and the slope in transversal direction is $u_x(t, x)$. Let us apply Newton's Law for the string from $x = x_0$ to $x = x_1$. Now applying the decomposition of the force, we have that on the longitudinal direction, the force is $\frac{T}{\sqrt{1+u_x}}$, which should be 0. The force on the transversal

direction is $\frac{Tu_x}{\sqrt{1+u_x}}$. By Newton's second law, it should also be mass times acceleration, which is $\rho \times u_{tt}$. Therefore, there is one equation for each direction of the decomposition of the force,

$$\frac{T}{\sqrt{1+u_x^2}}\Bigg|_{x_0}^{x_1} = 0, \tag{1.1}$$

$$\frac{Tu_x}{\sqrt{1+u_x^2}}\Bigg|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt} dx. \tag{1.2}$$

Looking at the binomial expansion for $\sqrt{1+u_x^2}$, the formula is

$$(1+u_x^2)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} u_x^{2n}. \tag{1.3}$$

When $|u_x|$, the slope of the string, is small enough, all the higher power terms can be considered as very small. Therefore, $(1+u_x^2)^{\frac{1}{2}}$ can be estimated as 1. The first equation then gives that $T(t,x)\Big|_{x_0}^{x_1} = 0$. Therefore, the tension does not depend on time and position. The second equation gives that $Tu_x\Big|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt} dx$. Differentiating both sides, this equation gives us that $(Tu_x)_x = \rho u_{tt}$. Since T does not depend on x, the form can be further simplified as $\frac{T}{\rho} u_{xx} = u_{tt}$. Let $c = \sqrt{\frac{T}{\rho}}$, then form of the wave equation, $u_{tt} = c^2 u_{xx}$, is derived.
The next step is to discover a way to solve the wave equation. In the proof of a very important formula to solve the wave equation, another important partial differential equation, transport equation, will be used. It is crucial to introduce the following concept and solving method related to transport equation first.

Definition. We call the partial differential equation with the form

$$u_t + c(x)u_x = 0$$

a transport equation with variable coefficient.

This equation is called transport equation because it can be used to model a fluid flowing at a constant rate along a one dimensional line. To solve this type of partial differential equation, one particular method, called the *method of characteristic curves*, is very useful. For the equation $u_t + c(x)u_x = 0$, first observing that such equation asserts that the directional derivative in the direction $(1, c(x))$ is zero. Therefore the curves along that direction must be constant. Solving this curve by solving the following ordinary differential equation

$$\frac{dx}{dt} = \frac{c(x)}{1}.$$

Using the fact that the curves along that direction must be constant and initial condition, the exact solution of the transport equation can be calculated. One example is necessary to demonstrate this method.

Example. Considering the following transport equation

$$\begin{cases} u_t - xu_x = 0, \\ u(0,x) = x. \end{cases}$$

Solving the characteristic curves, the differential equation is

$$\frac{dx}{dt} = \frac{-x}{1}.$$

By separation of variables, the solution is $x = ce^{-t}$. Since on this curve, the value of u is constant, the equation of u can be rewritten by $u(t, ce^{-t}) = u(0, c)$, where c can be rewritten as $c = xe^t$. Therefore it can be concluded that u is of the form $g(xe^t)$, where g is any function with one input. The final step is to use the initial condition to find this function g. Plugging in $t = 0$ and $u(0, x) = x$, the equation gives us $g(x) = x$. Thus the final solution for u is $u(t, x) = xe^t$.
Now with the necessary concept and method introduced, the general solution and the initial value problem of a wave equation can be found. The first theorem is related to the general solution for a one dimensional wave equation.

Theorem 1.1. *The general solution for a one-dimensional wave equation is*

$$u(t,x) = f(x + ct) + g(x - ct), \tag{1.4}$$

where f and g are two arbitrary (twice differentiable) functions of a single variable.

Proof. The general solution can be proven nicely because from our form of a one-dimension wave equation, the equation can be factored out nicely. The factorization is:

$$u_{tt} - c^2 u_{xx} = (\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})u = 0 \tag{1.5}$$

Set $v = u_t + cu_x$. Then the first part of the equation gives that $(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})v = 0$ Notice that this is a transport equation. The method of characteristics gives that the solution v is constant in the direction of the vector $V = i - cj$. The vector (c,1) is orthogonal to V. The lines that are parallel to V have the equations ct + x equals to some constant. Therefore, v(t,x) only depends on the value of t + cx. Then after plugging in $v = u_t + cu_x$, the equation becomes

$$u_t + cu_x = h(x + ct). \tag{1.6}$$

One particular solution for u is $f(x + ct)$ where $f'(s) = \frac{h(s)}{2c}$ because by chain rule, after differentiating, the result is $(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})f(x + ct) = \frac{ch(x+ct)}{2c} + \frac{ch(x+ct)}{2c} = h(x + ct)$. Since the equation is linear, by simple substitution, it can be checked that $f(x + ct) + g(x - ct)$ is also a solution for the wave equation. This is a particular solution plus the solution of the homogeneous equation of $u_t + cu_x = h(x + ct)$. Then it needs to be checked that indeed every solution of the wave equation has the form $f(x + ct) + g(x - ct)$. Let u to be any solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$. Then from the first half of the proof, it is clear that $u_t + cu_x = h(x + ct)$. Define a new function $w(x, t)$ to be

$$w(t,x) = u(t,x) + l(x + ct) - \frac{1}{2c}\int_{x-ct}^{0} h(s)ds \tag{1.7}$$

where l is just any smooth function. Notice that $w_t = u_t + cl'(x+ct) - \frac{1}{2}h(x-ct)$ and $w_x = u_x + l'(x+ct) + \frac{1}{2c}h(x-ct)$. Therefore, $w_t - cw_x = u_t + cl'(x+ct) - \frac{1}{2}h(x-ct) - cu_x - cl'(x+c) - \frac{1}{2}h(x-ct) = u_t - cu_x - h(x-ct) = h(x-ct) - h(x-ct) = 0$. Therefore, by the general solution of a transport equation, $w(t,x) = k(t+cx)$ for some smooth function k. As a result, $u(t,x) = f(x+ct) + g(x-ct)$, where $f(x+ct) = k(x+ct) - l(x+ct)$ and $g(x+ct) = \frac{1}{2c}\int_{x-ct}^{0} h(s)ds$. Since k, l, and h are arbitrary, this proves that every solution of the wave equation has the form $f(x+ct) + g(x-ct)$.
From the two parts, it can be concluded that the general solution for a one dimensional wave equation is $u(t,x) = f(x+ct) + g(x-ct)$. □

This proof is a detailed version of the proof that can be found in [6]. Next important theorem is about the solution of the initial value problem of the wave equation. The initial value problem is to solve the wave equation

$$u_{tt} = c^2 u_{xx}$$

with the initial conditions

$$u(0,x) = \phi(x)$$
$$u_t(0,x) = \psi(x),$$

where ϕ and ψ are just two arbitrary functions. It is well-known that there exists a unique solution for the initial value problem. The following result related to the solution of the initial value proven by d'Alembert in 1746, is very important. The proof is very short with the general solution of the wave equation proven. Therefore the proof of the d'Alembert formula is also included in this section.

Theorem 1.2. ***d'Alembert Formula*** *The solution formula for the initial-value problem of the one-dimensional wave equation*

$$u_{tt} = c^2 u_{xx}, -\infty < x < \infty, t > 0 \tag{1.8}$$

with the intial condition

$$u(0,x) = \phi(x)$$
$$u_t(0,x) = \psi(x)$$

is

$$u(t,x) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x)ds.$$

Proof. By the general solution of the wave equation, $u(t,x) = f(x+ct) + g(x-ct)$. Plugging in t = 0, the initial conditions shows that $\phi(x) = f(x) + g(x)$. Taking the derivative and plugging in $t = 0$, the initial conditions shows that $\psi(x) = cf'(x) - cg'(x)$. Also differentiating $\phi(x) = f(x) + g(x)$ and regard the two equations as two equations for two unknown functions. Therefore, the following system of equations is given

$$\phi' = f' + g'$$
$$\frac{1}{c}\psi = f' - g'.$$

In order to not confusing the variable name x with the original position x, set the variable name x to be s. The function of f' can be gotten by adding the two equations together and dividing the sum by 2. The function of g' can be gotten by subtracting the second equation from the first equation and dividing the sum by 2. In other words, the function of f' and g' are

$$f' = \tfrac{1}{2}(\phi' + \tfrac{\psi}{c}) \text{ and } g' = \tfrac{1}{2}(\phi' - \tfrac{\psi}{c}).$$

Integrating both equations, and this provides

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + C_1 \tag{1.9}$$

and

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + C_2, \tag{1.10}$$

where C_1 and C_2 are two constants. Now remember that $\phi(s) = f(s)+g(s)$ from the beginning of the proof, $C_1 + C_2 = 0$. Therefore, using the general solution of the wave equation, the solution for the initial value problem is just to substitute $s = x + ct$ for $f(s)$, and $s = x - ct$ for $g(s)$, then take the sum. Therefore, the solution is

$$u(t,x) = \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi. \tag{1.11}$$

Combining the two integral together, the formula can be further simplified to be the solution formula known as d'Alembert formula. The formula is

$$u(t,x) = \frac{1}{2}[(\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{1.12}$$

□

This is also a detailed version of the proof in [6]. Then the d'Alembert Formula can be used to derive the formula for inhomogeneous wave equation, which will be useful later in this paper for deriving results for accuracy.
Now using the solution formula for the initial-value problem of the homogeneous wave equation, the formula for inhomogeneous wave equation can also be derived.

Theorem 1.3. *Consider the one-dimensional inhomogeneous equation*

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(t,x), -\infty < x < \infty, t > 0, \\ u(0,x) = \phi(x), \\ u_t(0,x) = \psi(x). \end{cases} \tag{1.13}$$

The solution to the initial value problem is

$$u(t,x) = \frac{1}{2}[(\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)dyds. \tag{1.14}$$

Proof. This is an important result that will be used to discuss the accuracy of one of the method later in this paper. Using superposition principle, the original problem can be broken up into two simple equations:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, -\infty < x < \infty, t > 0, \\ u(0,x) = \phi(x), \\ u_t(0,x) = \psi(x), \end{cases} \tag{1.15}$$

and

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x), -\infty < x < \infty, t > 0, \\ u(0,x) = 0, \\ u_t(0,x) = 0. \end{cases} \tag{1.16}$$

Adding the solutions for both wave equations, the solution for the initial-value problem in the theorem will be proven. Equation (1.15) can be solved directly using theorem 1.2. The goal now is to look at equation (1.16). The idea of the proof of solution for equation (1.16) can be found in [3]. A more detailed version will be provided here. This proof uses the idea that comes from Duhamel's principle for Heat Equation. First of all, assume that $u = u(t,x,s)$ is the solution of the wave equation

$$\begin{cases} u_{tt}(\cdot;s) - c^2 u_{xx}(\cdot;s) = 0, \text{ in } (s,\infty) \times \mathbb{R}, \\ u(\cdot;s) = 0, u_t(\cdot;s) = f(s,\cdot), \text{ on } (t=s) \times \mathbb{R}. \end{cases} \tag{1.17}$$

and set

$$u(t,x) = \int_0^t u(t,x;s)ds \ (x \in \mathbb{R}^n, t \geq 0). \tag{1.18}$$

Formula (1.18) can be proven to be the solution of the equation (1.16). This can be proven by directly plugging in (1.18) in the the wave equation. Take the first derivative of (1.18), by Leibniz's rule, the result is

$$u_t(t,x) = u(t,x;t) + \int_0^t u_t(t,x;s)ds. \tag{1.19}$$

$u_t(0,x) = u(0,x;t) + 0 = 0$, which gives us the second initial condition in (1.16).

By the initial conditions for the wave equation (1.17), $u(t,x,t) = 0$, which means that $u_t(t,x) = \int_0^t u_t(t,x;s)ds$. Now taking the second derivative respect to t, and the result is that

$$u_{tt}(t,x) = u_t(t,x,t) + \int_0^t u_{tt}(t,x;s)ds. \tag{1.20}$$

Again from the initial conditions for (1.17), (1.20) becomes $u_{tt}(t,x) = f(t,x) + \int_0^t u_{tt}(t,x;s)ds$. Now looking at $u_{xx}(t,x;s)$, the result is that

$$u_{xx}(t,x) = \int_0^t u_{xx}(t,x;s)ds = \int_0^t \frac{1}{c^2} u_{tt}(t,x;s)ds. \tag{1.21}$$

Therefore, subtracting u_{tt} and c^2u_{xx}, the result is $u_{tt} - c^2u_{xx} = f(t,x) + \int_0^t u_{tt}(t,x;s)ds - \int_0^t u_{tt}(t,x;s)ds = f(t,x)$. Indeed $\int_0^t u(t,x;s)ds$ is a solution for the initial value problem (1.16). Now notice that the solution to the equation (1.17) can be obtained by simply using d'Alembert Formula. Therefore, $u(t,x;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(s,y)dy$. And the solution for (1.16) is then $u(t,x) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$. By superposition principle, by adding the solution from theorem 1.2 and this solution, indeed the solution form is as shown in (1.14). □

1.2 High Frequency Wave

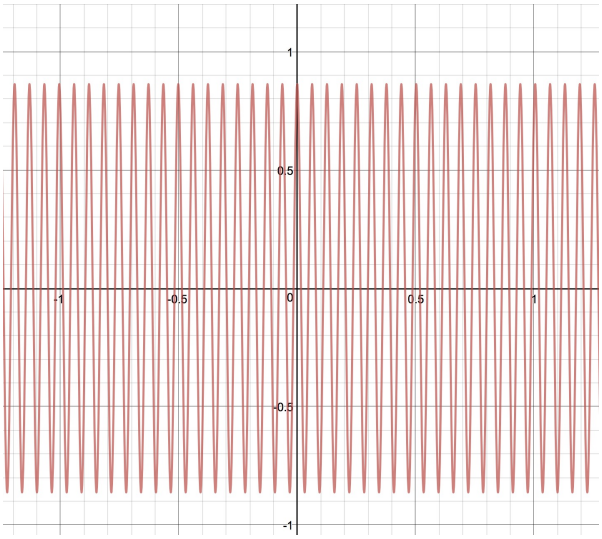
My research focuses on waves that have a very high frequency. Consider the following physical example. If we look at seismic waves, which are waves of energy that travel through the Earth's layers. There are two types of body waves, and one of them, the primary wave, can be modeled by a wave equation. Notice that the wave length of primary wave is at the level of $10 - 10^2$ m. The diameter of Earth is at the level of 10^3 km. Therefore, we know that the primary waves have a very short period. Thus the primary wave can be modeled by the following initial value problem

$$\begin{cases} u_{tt} - c(x)^2\Delta u = 0, \\ u(x,0) = A(x)e^{\frac{\mathrm{i}}{\epsilon}S(x)}, \\ u_t(x,0) = \frac{\mathrm{i}}{\epsilon}B(x)e^{\frac{\mathrm{i}}{\epsilon}S(x)}. \end{cases}$$

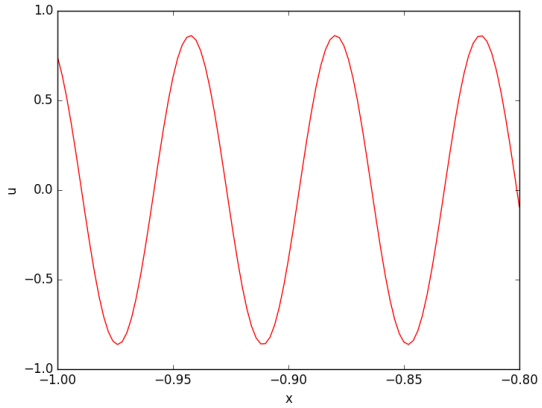
where A(x) is the amplitude of the initial wave, and S(x) is used to model each phase of the wave. Here ϵ is a very small number and $e^{\frac{i}{\epsilon}}$ is used to model the high frequency. Unlike the wave equation we have discussed in this paper earlier, the coefficient of this wave equation is a function of x, which is given and depends on the spring constant and the density of the medium. When x is more than one dimension, it is almost impossible to find an analytic solution of the above wave equation. If we try to solve this equation by using numerical methods, we will need to use many points to approximate the solution, which is very inefficient. Let us use the following easy example to demonstrate what will happen. Let us consider one-dimensional high frequency wave equation defined as below

$$\begin{cases} u_{tt} - u_{xx} = 0, \\ u(x,0) = e^{10^2\mathrm{i}x}, \\ u_t(x,0) = 0. \end{cases}$$

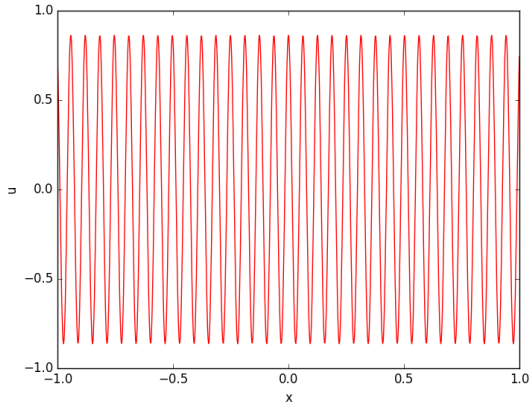
By d'Alembert's formula, we will have that the real part of the solution to be $Re(u(t,x)) = \frac{1}{2}\cos(10^2(x+t)) + \frac{1}{2}\cos(10^2(x-t))$. If we look at the wave at $t = 1$, then we will have a graph of equation $\frac{1}{2}\cos(10^2(x+1)) + \frac{1}{2}\cos(10^2(x-1))$, which has the graph



However, if we only plot uniformly 101 points in the interval [-1,1], we will have the following graph



Comparing to the actual solution, we notice that the result is not very accurate. Using 1000 points, and we will have



which is close to what the actual plot is. However, we need 1000 points just for the interval [-1,1]. Usually we want to modify the wave for a large interval, and the solution will not be as easy as our example. Thus we will need some new methods to simplify this wave equation first rather than directly solving the equation numerically.

2 Introduction

2.1 Asymptotic Approaches

The initial value problem for a high frequency wave equation

$$\begin{cases} u_{tt} - c(x)^2 \Delta u = 0, \\ u(x,0) = A(x) e^{\frac{\mathrm{i}}{\epsilon} S(x)}, \\ u_t(x,0) = \frac{\mathrm{i}}{\epsilon} B(x) e^{\frac{\mathrm{i}}{\epsilon} S(x)}. \end{cases}$$

cannot be solved analytically if x is more than one dimension. Moreover, it is also very inefficient to solve the equation using numerical methods. The aim of this paper is to find an efficient method to find the solution of the high frequency wave equation. This paper will discuss three methods to solve this kind of wave equations. They all have their weaknesses and strengths. The first method will be using the classic Wentzel-Kramers-Brillouin method. This method is easy to understand because the initial condition is used as a guess of the real solution. It involves the 'phase functions' and the 'amplitude functions'. In this method, all of the 'phase functions' and 'amplitude functions' can be found using simple first order ordinary differential equations and partial differential equations. The weakness of this method is that the asymptotic solution is invalid at shocks since the amplitude $A(t, x)$ does not have solution at the shocks. However, it is very important for people to determine the solutions around the caustics, especially for the case where people try to study the seismic waves.
The second method is the Gaussian beam method. The difference between the WKB method and the Gaussian beam method is that the Gaussian beam allows the phase function $S(t, x)$ to be complex off its center, and the imaginary part of the phase function is chosen so that the solution decays exponentially away from the center. The Lagrangian formulation consists of the ray tracing equations in the WKB method. The validity of the construction is presented by Ralston in [2]. Since the estimation has an imaginary part, the 'width' of the estimation prevents the formulation of a shock. The construction of Gaussian beam approximation is based on the truncation of the Taylor expansion of the phase function around the beam center up to the second order. Therefore, when the width of the beam becomes large, the method will lose accuracy. This happens when the solution of the wave equations spreads, exactly the opposite of what makes the WKB method bad. Certainly, one way to solve this by doing reinitialization once in a while. This, however, increases the computational complexity when the beams spread quickly.
The third method, therefore, is to modify the Gaussian functions by fixing the width. This is where the 'frozen' comes from in the name frozen Gaussian approximation. In FGA, the solution is approximation by a superposition of Gaussian functions, living in the phase space, and each function is not necessary an asymptotic solution. The FGA can overcome the problem of spreading solution by fixing the width, and it will also be accurate around a shock where the WKB method will break.

This paper is based on several papers about those methods. The aim is to discuss and compare all three methods, and to conclude that under what kind of restraints these methods are valid.

3 Wentzel-Kramers-Brillouin Method

3.1 Motivation and Derivation

The following initial value problem of the wave equation:

$$\begin{cases} u_{tt} - c(x)^2 \Delta u = 0, \\ u(x,0) = A(x) e^{\frac{\mathrm{i}}{\epsilon} S(x)}, \\ u_t(x,0) = \frac{\mathrm{i}}{\epsilon} B(x) e^{\frac{\mathrm{i}}{\epsilon} S(x)}. \end{cases} \tag{3.1}$$

needs to be solved, where ϵ is a very small number. The following ansatz of the general solution of Wentzel-Kramers-Brillouin method will be considered,

$$u(t,x) = A(t,x) e^{\frac{\mathrm{i}s(t,x)}{\epsilon}}, \tag{3.2}$$

where A(t,x) is the amplitude function that depends on the time and the disposition x. S(t,x) is the phase function that also depends on time and the disposition. Notice that the guess of the solution is the similar to the form of the initial condition, except this time the functions A and S are functions of t and x instead of just x. plugging in the approximation into the wave equation, the calculation is as the following:

$$u_t = A_t(t,x) e^{\frac{\mathrm{i}s(t,x)}{\epsilon}} + A(t,x) \frac{\mathrm{i}}{\epsilon} s_t(t,x) e^{\frac{\mathrm{i}s(t,x)}{\epsilon}}, \tag{3.3}$$

$$u_{tt} = A_{tt} e^{\frac{\mathrm{i}s}{\epsilon}} + \frac{2\mathrm{i}}{\epsilon} A_t s_t e^{\frac{\mathrm{i}s}{\epsilon}} + \frac{\mathrm{i}}{\epsilon} A s_{tt} e^{\frac{\mathrm{i}s}{\epsilon}} - A \frac{1}{\epsilon^2} (s_t)^2 e^{\frac{\mathrm{i}s}{\epsilon}}. \tag{3.4}$$

and for u_{xx} we can simply replace all t with x, and the result is

$$u_{xx} = A_{xx} e^{\frac{\mathrm{i}s}{\epsilon}} + \frac{2\mathrm{i}}{\epsilon} A_x s_x e^{\frac{\mathrm{i}s}{\epsilon}} + \frac{\mathrm{i}}{\epsilon} A s_{xx} e^{\frac{\mathrm{i}s}{\epsilon}} - A \frac{1}{\epsilon^2} (s_x)^2 e^{\frac{\mathrm{i}s}{\epsilon}}. \tag{3.5}$$

Now plug both u_{tt} and u_{xx} in the wave equation, the wave equation gives

$$\begin{aligned} & u_{tt} - c^2(x) u_{xx} = 0, \\ & A_{tt} e^{\frac{\mathrm{i}s}{\epsilon}} + \frac{2\mathrm{i}}{\epsilon} A_t s_t e^{\frac{\mathrm{i}s}{\epsilon}} + \frac{\mathrm{i}}{\epsilon} A s_{tt} e^{\frac{\mathrm{i}s}{\epsilon}} - A \frac{1}{\epsilon^2} (s_t)^2 e^{\frac{\mathrm{i}s}{\epsilon}} \\ & - c^2(x) (A_{xx} e^{\frac{\mathrm{i}s}{\epsilon}} + \frac{2\mathrm{i}}{\epsilon} A_x s_x e^{\frac{\mathrm{i}s}{\epsilon}} + \frac{i}{\epsilon} A s_{xx} e^{\frac{\mathrm{i}s}{\epsilon}} - A \frac{1}{\epsilon^2} (s_x)^2 e^{\frac{\mathrm{i}s}{\epsilon}}) = 0, \end{aligned}$$

assorting the terms based on the order of ϵ in the denominator of the coefficients, the following three equations are formed, ordered by the contributions to the solution:

$$(c^2 A (s_x)^2 - A (s_t)^2) = 0, \tag{3.6}$$

$$2(A_t s_t - c^2 A_x s_x) + A(s_{tt} - c^2 s_{xx}) = 0, \tag{3.7}$$

$$A_{tt} - c^2 A_{xx} = 0, \tag{3.8}$$

where the solution of s and A will be given from the first two equations. The third equation is not useful for solving s and A because A is the guessed solution, which has error terms. However, the third equation is useful when calculating the accuracy of this method.
From the first equation, the following equations are given:

$$(cs_x)^2 - (s_t)^2 = 0,$$
$$(cs_x - s_t)(cs_x + s_t) = 0.$$

Therefore, either of $(cs_x - s_t) = 0$ or $(cs_x + s_t) = 0$ would provide one solution for s. This is a Hamiltonian system, which means that solving either of the equation will provide the final solution of the wave equation. Picking $cs_x - s_t = 0$ will simplify the later calculations. Notice that this is a transport equation, which can be solved using the method of characteristics. Using $c(x)s_x = s_t$ and the result that $c(x)s_{xt} = s_{tt}$ and $c'(x)s_x + c(x)s_{xx} = s_{xt}$, the second equation gives:

$$\begin{aligned}
&2(A_t s_t - c^2 A_x s_x) + A(s_{tt} - c^2 s_{xx}) = 0,\\
&2(A_t s_t - cA_x s_t) + A(cs_{xt} - cs_{xt} - c'(x)cs_x) = 0,\\
&2A_t - cA_x - Ac'(x) = 0,\\
&2A_t - (cA)_x = 0,\\
&\frac{2}{c(x)}(cA)_t = (cA)_x,
\end{aligned}$$

which again can be solved using the method of characteristics and will provide the function for the amplitude function $A(t, x)$. Therefore, since both A and s are solved, the approximation of the solution for the wave equation is obtained.

3.2 Accuracy

One of the important things to look at for a method is the accuracy. For this example, the order of the error should be at least $\mathrm{O}(\epsilon)$ because otherwise the error goes to 0 slower than ϵ goes to 0, and then there is really no point to use asymptotic method because the ϵ for the asymptotic method is always very small. The following theorem will be proven in this section.

Theorem 3.1. *The error of the WKB method to solve the high frequency wave equation should be at least O(ϵ).*

Proof. First of all, set error term to be $R^\epsilon(t,x) = \epsilon A_1(t,x)\,\mathrm{e}^{\frac{\mathrm{i}}{\epsilon}s(t,x)} + r^\epsilon(t,x)$, where $\epsilon A_1(t,x)e^{\frac{\mathrm{i}}{\epsilon}s(t,x)}$ is a correction term that will make the calculation work. Thus the real solution is in the form

$$u(t,x) = (A_0(t,x) + \epsilon A_1(t,x))e^{\frac{\mathrm{i}s}{\epsilon}} + r^\epsilon(t,x).$$

Plugging in this real solution into the wave equation, and the following equation is true,

$$\begin{aligned}
&(A_{0tt} + \frac{2\mathrm{i}}{\epsilon}A_{0t}s_t + \frac{\mathrm{i}}{\epsilon}A_0 s_{tt} - A_0\frac{1}{\epsilon^2}(s_t)^2 + \epsilon A_{1tt} + 2iA_{1t}s_t + \mathrm{i}A_1 s_{tt}\\
&- A_1\frac{\mathrm{i}}{\epsilon}(s_t)^2 - \epsilon^2(x)(A_{0xx} + \frac{2\mathrm{i}}{\epsilon}A_{0x}s_x + \frac{\mathrm{i}}{\epsilon}A_0 s_{xx} - A_0\frac{1}{\epsilon^2} + \epsilon A_{1xx}\\
&+ 2\mathrm{i}A_{1x}s_x + \mathrm{i}A_1 s_{xx} - A_1\frac{1}{\epsilon}(s_x)^2))e^{\frac{\mathrm{i}s}{\epsilon}} = r^\epsilon_{xx}(t,x) - r^\epsilon_{tt}(t,x).
\end{aligned}$$

Now rearranging the terms based on the order of ϵ, since ϵ is very small, all the terms with the same order of ϵ must be 0. The four equations are:

$O(\frac{1}{\epsilon^2})$:

$$-A_0\frac{\mathrm{i}}{\epsilon^2}(s_t)^2 - c^2(x)(-A_0\frac{\mathrm{i}}{\epsilon}(s_x)^2) = 0. \tag{3.9}$$

$O(\frac{1}{\epsilon})$:

$$\frac{2\mathrm{i}}{\epsilon}A_{0t}s_t + \frac{\mathrm{i}}{\epsilon}A_0 s_{tt} - A_1\frac{\mathrm{i}}{\epsilon}(s_t)^2 - c^2(x)(\frac{2\mathrm{i}}{\epsilon}A_{0x}s_x + \frac{\mathrm{i}}{\epsilon}A_0 s_{xx} - A_1\frac{\mathrm{i}}{\epsilon}(s_x)^2) = 0. \tag{3.10}$$

$O(1)$:

$$A_{0tt} + 2\mathrm{i}A_{1t}s_t + \mathrm{i}A_1 s_{tt} - c^2(x)(A_{0xx} + 2\mathrm{i}A_{1x}s_x + \mathrm{i}A_1 s_{xx}) = 0. \tag{3.11}$$

$O(\epsilon)$:

$$(\epsilon A_{1tt} - c^2(x)(\epsilon A_{1xx}))e^{\frac{is}{\epsilon}} + r^\epsilon_{tt}(t,x) - r^\epsilon_{xx}(t,x) = 0. \tag{3.12}$$

From equation (3.3), since $A_{0tt} - c^2(x)A_{0xx} = 0$, the equation becomes $2\mathrm{i}(A_{1t}s_t - c^2(x)A_{1x}s_x) + \mathrm{i}(A_1 s_{tt} - c^2(x)A_1 s_{xx}) = 0$. Notice that this form is similar to what has been calculated for the approximated solution $A_0(t,x)$, which satisfies $2(A_{0t}s_t - c^2 A_{0x}s_{0x}) + A(s_{tt} - c^2 s_{xx}) = 0$. Therefore, A_1 has order at least the same as A_0, which is at least $O(1)$.
Next thing to look at is equation (3.4). The equation can be rewritten as $r^\epsilon_{tt} - r^\epsilon_{xx} = -\epsilon(A_{1tt} - c^2(x)A_{1xx})e^{\frac{is}{\epsilon}}$. This is an inhomogenious wave equation, which has the form

$$\begin{cases} r^\epsilon_{tt} - r^\epsilon_{xx} = -\epsilon(A_{1tt} - c^2(x)A_{1xx})e^{\frac{is}{\epsilon}}, \\ r^\epsilon(0,x) = 0, \\ r^\epsilon_t(0,x) = 0, \end{cases} \tag{3.13}$$

where the initial conditions are 0 because the initial conditions for the initial equation were used to approximate the solution, so at the initial conditions the error terms are 0. Now using theorem 1.3 proved in introduction, the solution will have the form $r^\epsilon(t,x) = \frac{1}{2c}\int_0^t \int_{x-c(t-p)}^{x+c(t-p)} -\epsilon(A_{1pp} - c^2(y)A_{1yy})e^{\frac{\mathrm{i}s(p,y)}{\epsilon}}\,dydp$, since A_1 has order O(1), this integral gives that r^ϵ has order $O(\epsilon)$. Now by looking at the total error term $R^\epsilon(t,x) = \epsilon A_1(t,x)e^{\frac{\mathrm{i}}{\epsilon}s(t,x)} + r^\epsilon(t,x)$, since $A_1(t,x)$ has order $O(1)$ and $r^\epsilon(t,x)$ has order $O(\epsilon)$, the error term $R^\epsilon(t,x)$ also has order $O(\epsilon)$.

□

3.3 Weakness

Using the classical WKB method in multi-dimensional cases, by similar grouping as the first dimensional case illustrate in this paper, the solution will be

$$s_t + \frac{1}{2}|\Delta s|^2 = 0, \tag{3.14}$$

$$\rho_t + \Delta\cdot(\rho\Delta S) = 0. \tag{3.15}$$

This equation is Hamiltonian-Jacobi type, and the solution becomes singular near the caustics. Therefore, this method will not give accurate solutions near caustics. However, in some applications, and in seismic waves in particular, people want to know the accurate solution near the caustics. Therefore, a better method is needed to solve the high frequency wave equation.

4 Gaussian Beam Method

4.1 Motivation

As the previous section shows, although WKB is an easy way to approximate the solution, sometimes people really need an accurate solution near the caustics, which the WKB method cannot provide. The Gaussian beam method, another method for high frequency wave equation, can be very efficient and also allows accurate computation near the caustics. Similar to WKB method, the Gaussian beam also has a WKB form $u(t,x) = A(t,x)e^{\frac{is(t,x)}{\epsilon}}$. The main difference is that by adding some extra terms into the ansatz, the phase function $s(t,x)$ can be complex off the center. The imaginary part of the phase function is chosen so that like a normal distribution the solution decays exponentially away from the center. The 'width' the imaginary part adds to the approximation make the solution valid at caustics. The detailed analysis can be found by Ralston's paper in [2].
Lagrangian numerical methods of Gaussian beams were usually developed based on Taylor expansion and superposition principle, and the accuracy of the beam off the center is usually determined by the truncation error of the Taylor expansion. The accuracy of the Taylor expansion is given by was studied by Motamed and Runborg [4], and Tanushev [5] developed and analyzed higher order Gaussian beams giving better accuracy of the approximations.

4.2 Gaussian Beam Method

In this section, the Gaussian beam method will be calculated. The following ansatz will be used

$$u(t,x,y) = A(t,y)e^{\mathrm{i}T(t,x,y)/\epsilon}, \tag{4.1}$$

where $y = y(t,y_0)$, and $T(t,x,y)$ is given by the Taylor expansion

$$T(t,x,y) = s(t,y) + p(t,y)(x-y) + \frac{1}{2}M(t,y)(x-y)^2. \tag{4.2}$$

This ansatz has a similar form as the WKB method, except a free parameter y is used to control the domain where the WKB method is applied. y can be considered as the beam center, chosen by the following equations

$$\frac{dy}{dt} = p(t,y), y(0) = y_0, \tag{4.3}$$

and the ansatz is called beam-shaped ansatz.
The goal is to first find the equations for A, s, p, and M. Just like before, plug this form into the wave equation. the equation will give us the result

$$u_t = A_t + A\frac{\mathrm{i}}{\epsilon}(s_t + p_t(x-y) - p\frac{dy}{dt} + \frac{1}{2}M_t(x-y)^2 - \frac{dy}{dt}M(x-y)$$
$$\exp(\frac{\mathrm{i}}{\epsilon}(s(t,y) + p(t,y)(x-y) + \frac{1}{2}M(t,y)(x-y)^2)).$$

Taking another derivative, u_{tt} is given by

$$\begin{aligned}u_{tt} = \{&A_{tt} + 2A_t\frac{i}{\epsilon}(s_t + p_t(x-y) - p\frac{dy}{dt} + \frac{1}{2}M_t(x-y)^2 - \frac{dy}{dt}M(x-y)\\
&+ A\frac{\mathrm{i}}{\epsilon}(s_{tt} + p_{tt}(x-y) - 2p_t\frac{dy}{dt} - p\frac{d^2y}{dt^2} + \frac{1}{2}M_{tt}(x-y)^2 - 2M_t\frac{dy}{dt}(x-y)\\
&- \frac{d^2y}{dt^2}M(x-y) + (\frac{dy}{dt})^2M) - A\frac{1}{\epsilon^2}(s_t + p_t(x-y) - p\frac{dy}{dt} + \frac{1}{2}M_t(x-y)^2 - \frac{dy}{dt}M(x-y))^2\}\\
&\exp(\frac{\mathrm{i}}{\epsilon}(s(t,y) + p(t,y)(x-y) + \frac{1}{2}M(t,y)(x-y)^2)).\end{aligned}$$

Next, taking derivatives respect to x. This is shorter than the u_{tt} term because y is not a function of x, and the following result is given

$$u_x = A\frac{\mathrm{i}}{\epsilon}(p + M(x-y))\exp(\frac{\mathrm{i}}{\epsilon}(s(t,y) + p(t,y)(x-y) + \frac{1}{2}M(t,y)(x-y)^2)).$$

and u_{xx} is in the following term:

$$u_{xx} = (A\frac{i}{\epsilon}M + A - \frac{1}{\epsilon^2}(p + M(x-y))^2)\exp(\frac{i}{\epsilon}(s(t,y) + p(t,y)(x-y) + \frac{1}{2}M(t,y)(x-y)^2)).$$

Just like calculating the Wentzel-Kramers-Brillouin method, let us order the terms. This time, however, the order of (x-y) should also be taken into account. For $c^2(x)$, it is necessary to get terms $(x-y)$ by Taylor's expansion, The following Taylor expansion is given for $c^2(x)$,

$$c^2(x) = c^2(y) + (c^2(y))_y(x-y) + \frac{1}{2}(c^2(y))_{yy}(x-y)^2 + \ldots + \frac{1}{n!}\frac{\partial^n c^2(y)}{\partial y^n}(x-y)^n, \qquad (4.4)$$

where the first n-th Taylor's expansion terms to approximate the function $c^2(x)$ using y is used. This time the terms are ordered by both the order of ϵ and the order of $(x-y)$. First, looking at the terms with ϵ^{-2} and $(x-y)^0$. Order the terms, and we cancel out the common term $\exp(\frac{i}{\epsilon}(s(t,y) + p(t,y)(x-y) + \frac{1}{2}M(t,y)(x-y)^2))$, the following equation is given

$$-A\frac{1}{\epsilon^2}(s_t - p_t\frac{dy}{dt})^2 - c^2(y)(A(-\frac{1}{\epsilon^2})p^2) = 0,$$

,where in this case $c^2(y)$ is used to approximate $c^2(x)$. Further simplifying this equation, the following equation is given

$$\begin{aligned}(s_t - p\frac{dy}{dt})^2 - c^2(y)p^2 &= 0,\\
s_t - p\frac{dy}{dt} &= \pm c(y)p.\end{aligned}$$

Like WKB method, using either sign will give the solution. For simplicity, choosing $s_t - p\frac{dy}{dt} = c(y)p$. Setting $\frac{dy}{dt} = -c(y)$, and this gives that $s_t = 0$. Next, grouping all the terms with $\frac{1}{\epsilon^2}(x-y)$, expanding the squares, and the following results will be given

$$\begin{aligned}&- A\frac{1}{\epsilon^2}(2s_tp_t(x-y) - 2p\frac{dy}{dt}p_t(x-y) - 2s_t\frac{dy}{dt}M(x-y)\\
&+ 2p(\frac{dy}{dt})^2M(x-y)) - ((c^2(y))_yA(-\frac{1}{\epsilon^2})p^2 - c^2(y)A(-\frac{1}{\epsilon^2})2pM)(x-y) = 0.\end{aligned}$$

Cancelling out the common terms $-A\frac{1}{\epsilon^2}$ and (x-y), the following equation is given:

$$2s_t p_t - 2p\frac{dy}{dt}p_t - 2s_t\frac{dy}{dt}M + 2p(\frac{dy}{dt})^2 M - (c^2(y))_y p^2 - c^2(y)2pM = 0.$$

Plugging in $s_t = 0$, the final equation is in the form

$$-2p\frac{dy}{dt}p_t + 2p(\frac{dy}{dt})^2 M - (c^2(y))_y p^2 - c^2(y)2pM = 0.$$

From the previous step, since $c(y)$ is set to satisfy $\frac{dy}{dt} = -c(y)$, which means that $(\frac{dy}{dt})^2 = c^2(y)$. plugging this back into the equation, we can further simplify the equation, and the equation for p_t will be given by:

$$\begin{aligned}
2pc(y)p_t &= (c^2(y))_y p^2, \\
p_t &= \frac{(c^2(y))_y}{2c(y)}p, \\
p_t &= \frac{2c(y)(c(y))_y}{2c(y)}p, \\
p_t &= (c(y))_y p.
\end{aligned}$$

At last, if looking at the terms with $\epsilon^{-2}(x-y)^2$, the following equation is given:

$$\begin{aligned}
&- A\frac{1}{\epsilon^2}(-2p_t\frac{dy}{dt}M(x-y)^2 + s_t M_t(x-y)^2 - p\frac{dy}{dt}M_t(x-y)^2 + (p_t)^2(x-y)^2 + (\frac{dy}{dt}M)^2(x-y)^2) \\
&- (A(-\frac{1}{\epsilon^2}))(c^2(y)M^2 + (c^2(y))_y 2pM + \frac{1}{2}(c^2(y))_{yy}p^2)(x-y)^2.
\end{aligned}$$

Again deleting all the common terms, the simplified equation is in the form:

$$-2p_t\frac{dy}{dt}M + s_t M_t - p\frac{dy}{dt}M_t + p_t^2 + (\frac{dy}{dt}M)^2 - c^2(y)M^2 - (c^2(y))_y 2pM - \frac{1}{2}(c^2(y))_{yy}p^2 = 0.$$

Plugging in $p_t = (c(y))_y p$, $\frac{dy}{dt} = -c(y)$ from the previous step, and $s_t = 0$, the following equation is given:

$$\begin{aligned}
&- 2(c(y))_y p(-c(y))M - p(-c(y))M_t + ((c(y))_y p)^2 + (-c(y)M)^2 - (c^2(y)M)^2 - (c^2(y))_y 2pM \\
&- \frac{1}{2}(c^2(y))_{yy}p^2 = 0, \\
&2p(c(y))_y c(y)M + pc(y)M_t + ((c(y))_y p)^2 - (c^2(y))_y 2pM - \frac{1}{2}(c^2(y))_{yy}p^2 = 0, \\
&2p(c(y))_y c(y)M - (c^2(y))_y 2pM + pc(y)M_t + ((c(y))_y p)^2 - ((c(y))_y)^2 p^2 = 0, \\
&2p(c(y))_y c(y)M - 4(c(y))_y c(y)pM + pc(y)M_t = 0, \\
&2p(c(y))_y c(y)M = pc(y)M_t, \\
&2(c(y))_y M = M_t.
\end{aligned}$$

Therefore the equation for M has also been calculated.
At last, grouping all the terms with $O(\epsilon^{-1}(x-y)^0)$, the equation for A will be given.

$$2A_t \cdot \frac{ds}{dt} - 2pA_t\frac{dy}{dt} + A(\frac{ds}{dt})^2 - 2A(\frac{dp}{dt})(\frac{dy}{dt}) - pA(\frac{dy}{dt})^2 - A \cdot M \cdot c^2(y) + A \cdot M \cdot (\frac{dy}{dt})^2 = 0.$$

Further simplifying this equation by plugging in all the equations gotten from the above, specifically $(\frac{dy}{dt})^2 = c^2(y)$ and $\frac{dp}{dt} = (c(y))_y p$, the equation becomes

$$2pc(y)A_t + 2A \cdot (c(y))_y \cdot p \cdot c(y) - p \cdot A \cdot (c(y))_y = 0.$$

Combining the last two terms in the equation, and express A_t in terms of A and $c(y)$, the solution will be

$$A_t = -\frac{Ac(y)_y}{2}.$$

Therefore, after solving for all the values, the correct approximation can be expressed as

$$u(t, x, y) = A(t, y)e^{iT(t,x,y)/\epsilon},$$

where $T(t, x, y) = s(t, y) + p(t, y)(x - y) + \frac{1}{2}M(t, y)(x - y)^2$. And the function A, s, M, and p are given by the ordinary differential equations

$$\begin{cases} s_t = 0, \\ p_t = (c(y))_y p, \\ M_t = 2(c(y))_y M, \\ A_t = -\frac{Ac(y)_y}{2}, \end{cases}$$

where $c(y) = -\frac{dy}{dt}$, and $\frac{dy}{dt}$ itself satisfies $\frac{dy}{dt} = p$. Therefore, the Gaussian beam method has been calculated.

4.3 Weakness

The Gaussian beam method is slightly more complicated than WKB method. By making the phase function complex off its center, it successfully overcome the problem that WKB cannot provide accurate solution near caustics. The truncation error of the Taylor expansion determines the accuracy of the beam because they are developed by Taylor expansion. The accuracy of the Taylor expansion was studied by Motamed and Runborg [4], and Tanushev [5]. Increasing the order of the beams will increase the accuracy of the approximation. This method, however, still has its weaknesses. From the derivation above, the truncation error used in the calculation of this paper is up to quadratic term, hence, it loses accuracy when the width of the beam gets large. In other words, the imaginary part of $M(t, y)$ in $T(t, x, y)$ becomes too small that the Gaussian function is not localized anymore. Exactly opposite to the reason why WKB is not accurate, this happens when the wave equation spreads. The following example from [1] is a good example to demonstrate the kind of equation that the Gaussian beam method will not provide accurate solution.

Example. Looking at the wave equation with speed $c(x) = x^2$. The initial conditions are:

$$u(0, x) = e^{-100(x-0.5)^2} e^{\frac{ix}{\epsilon}} \tag{4.5}$$

$$u_t(0, x) = -\frac{ix^2}{\epsilon} e^{-100(x-0.5)^2} e^{\frac{ix}{\epsilon}}. \tag{4.6}$$

Let the final time to be T = 0.5. And comparing the true solution to the solution solving by the Gaussian Beam Method using $\epsilon = \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$. In Dr. JiangFeng Lu and Dr. Xu Yang's paper, they used the finite difference method using the mesh size of $\delta_x = \frac{1}{2^{12}}$ and the time step $\delta_t = \frac{1}{2^{18}}$ to approximate the real solution. Notice that d'Alembert's formula proven in the background chapter will not apply here because the speed is not constant c but c(x). Therefore, the real solution is obtained by another numerical method. The ODEs of the Gaussian Beam Method is solved by using the time step $\delta_t = \frac{1}{2^{11}}$ and the mesh size $\delta_x = \frac{1}{2^{12}}$. Looking at the ℓ_2 and ℓ_∞, the errors are

ϵ	$\frac{1}{2^6}$	$\frac{1}{2^7}$	$\frac{1}{2^8}$
ℓ_∞	7.15×10^{-1}	5.08×10^{-1}	3.36×10^{-1}
ℓ_2	3.26×10^{-1}	2.28×10^{-1}	1.47×10^{-1}.

Notice that the rate of convergence is less than ϵ for both norms. This can be found out by dividing the next error by the previous error, and compare it with $\frac{1}{2}$. Therefore, the Gaussian Beam method is not good for some wave equations. A slightly better method should be introduced to overcome the shortcomings of the Gaussian Beam method.

5 Frozen Gaussian Beam Method

5.1 Motivation

As mentioned in the previous section, the Gaussian Beam Method uses a complex phase function

$$s(t, x, y) = s(t, y) + p(t, y) \cdot (x - y) + \frac{1}{2} M(t, y)(x - y)^2.$$

This method has difficulty getting accurate solution for the wave equations that spread. The new approximation, instead of using the regular Gaussian Beam Method that might spread overtime, will fix the 'width' of the beam, and the method is called Frozen Gaussian Beam method. As Dr. JianFeng Lu and Xu Yang mentioned in their paper, the waves can be decomposed into several branches of propagation, and each of them is approximated by Gaussian functions on phase plane. This method, therefore, is fundamentally different and more complicated than the Gaussian Beam Method because rather than making the initial data to be Gaussian, the solution of the wave equation is approximated by the Gaussian functions living in the phase space, and each one is not necessary an asymptotic solution.

5.2 Frozen Gaussian Beam Method

In this section, the Frozen Gaussian Beam Method will be calculated. The calculation of the Frozen Gaussian Beam Method is tricky because directly plugging in the ansatz will not provide the ODES to solve this approximation. The formulation of the Frozen Gaussian Beam method is as the following:
The approximation is by the integral representation,

$$u(t, x) = \frac{1}{2\pi\epsilon} \int \int \int a(t, q, p) e^{\frac{\mathrm{i}}{\epsilon} T(t,x,y,q,p)} dy dq dp,$$

where the T has the form

$$T(t,x,y,q,p) = s(t,q,p) + P(t,q,p)(x - Q(t,q,p)) + \frac{\mathrm{i}}{2}|x - Q(t,q,p)|^2 - p(y-q) + \frac{\mathrm{i}}{2}|y-q|^2.$$

The evolution of Q and P are given by the motion corresponding to the Hamiltonian

$$\begin{cases} \frac{dQ}{dt} = -c(Q), \\ \frac{dP}{dt} = \partial_Q c P, \end{cases}$$

with initial condition $\begin{cases} Q(0,q,p) = q, \\ P(0,q,p) = p, \\ s(0,q,p) = 0. \end{cases}$

The first instinct will be to use the same way that was used to solve the WKB and Gaussian beam method. Plugging in this approximated solution into the wave equation directly, and the result for u_{tt} is

$$\begin{aligned} u_t &= a_t e^{\frac{\mathrm{i}}{\epsilon}T} + a\frac{\mathrm{i}}{\epsilon}T_t e^{\frac{\mathrm{i}}{\epsilon}T}, \\ u_{tt} &= a_{tt}e^{\frac{\mathrm{i}}{\epsilon}} + 2a_t\frac{\mathrm{i}}{\epsilon}T_t e^{\frac{\mathrm{i}}{\epsilon}T} + a\frac{\mathrm{i}}{\epsilon}T_{tt}e^{\frac{\mathrm{i}}{\epsilon}T} + a(\frac{\mathrm{i}}{\epsilon})^2(T_t)^2 e^{\frac{\mathrm{i}}{\epsilon}\epsilon T}. \end{aligned}$$

Calculating the u_{xx}, the form becomes:

$$\begin{aligned} u_x &= a\frac{\mathrm{i}}{\epsilon}T_x e^{\frac{\mathrm{i}}{\epsilon}T}, \\ u_{xx} &= a\frac{\mathrm{i}}{\epsilon}T_{xx}e^{\frac{\mathrm{i}}{\epsilon}T} + a(\frac{\mathrm{i}}{\epsilon})^2(T_x)^2 e^{\frac{\mathrm{i}}{\epsilon}T}. \end{aligned}$$

Also, the form of T_t, T_{tt}, T_x, and T_{xx} will also be calculated. They are:

$$\begin{aligned} T_t &= s_t + P_t(x-Q) - PQ_t - Q_t\mathrm{i}(x-Q), \\ T_{tt} &= s_{tt} + P_{tt}(x-Q) + P_tQ_t(x-Q) - P_tQ_t - PQ_{tt} + \mathrm{i}(Q_t)^2 - Q_{tt}\mathrm{i}(x-Q), \\ T_x &= P + \mathrm{i}(x-Q), \\ T_{xx} &= \mathrm{i}. \end{aligned}$$

The way to deal with $c^2(x)$ is again using Taylor Expansion. The Taylor Expansion has the following form:

$$c^2(x) = c^2(Q) + (c^2(Q))_Q(x-Q) + \frac{1}{2}(c^2(Q))_{QQ}(x-Q)^2 + \dots.$$

Plugging these terms in the wave equation $u_{tt} = c^2(x)u_{xx}$, and grouping all the terms by the order of ϵ and $(x-Q)$. For $\epsilon^{-2}(x-Q)^0$, the equation is:

$$-s_t^2 + 2s_tPQ_t - (PQ_t)^2 + c^2(Q)\cdot P^2 = 0.$$

Since from the given Hamiltonian, $Q_t = -c(Q)$, $-(PQ_t)^2 = -c^2(Q)P^2$. The equation then becomes

$$-s_t(s_t - 2PQ_t) = 0.$$

This provides two sets of solutions. One solution is $s_t = 0$, and using this s_t will simplify the calculations for the rest of the unknown functions because many terms will simply be zero because of this term. Now looking at terms with order $\epsilon^{-2}(x-Q)^1$, grouping the terms, and the equation has the form:

$$\begin{aligned}
&-(-2P_tPQ_t + 2P(Q_t)^2 i) + (c^2(Q)\cdot 2\cdot P\cdot i + P^2(c^2(Q))_Q) = 0,\\
&2P_tQ_t - 2(Q_t)^2\mathrm{i} + c^2(Q)\cdot 2\cdot \mathrm{i} + P(c^2(Q))_Q = 0.
\end{aligned}$$

By chain rule, taking the derivative of $(c^2(Q))_Q$, and the result is $2c(Q)c(Q)_Q$, which will be denoted as $2cc_Q$. The result then can be further simplified as:

$$2P_tQ_t - 2(Q_t)^2\mathrm{i} + 2c^2\mathrm{i} + 2cPc_Q = 0.$$

Again, plugging in the Hamiltonian $Q_t = -c$, and the result is

$$\begin{aligned}
&-2P_tc - 2c^2\mathrm{i} + 2c^2\mathrm{i} + 2cPc_Q = 0,\\
&Pc_Q - P_t = 0,\\
&P_t = P\cdot\partial_Q c.
\end{aligned}$$

Therefore, the equation for P has also been calculated. This coincides with the Hamiltonian given in the problem. Next, from the terms with order $\epsilon^{-1}(x-Q)^0$ and $\epsilon^{-1}(x-Q)^2$, the function for a can also be calculated. First of all, for $\epsilon^{-1}(x-Q)^0$, arranging the terms, the equation is:

$$\begin{aligned}
&2a_tT_t + aT_{tt} - c^2(Q)\cdot a\cdot T_{xx} = 0,\\
&2a_ts_t - 2a_TPQ_t + as_{tt} - aP_tQ_t - aPQ_{tt} + \mathrm{i}a(Q_t)^2 - c^2(Q)a\mathrm{i} = 0.
\end{aligned}$$

Plugging in $s_t = 0$ to cancel out some terms, and the equation becomes

$$-2a_tPQ_t - aP_tQ_t - aPQ_{tt} + \mathrm{i}a(Q_t)^2 - c^2(Q)a\mathrm{i} = 0.$$

Then plugging in Hamiltonian $P_t = \partial_Q cP$ and $Q_t = -c(Q)$, the equation can be simplified as

$$\begin{aligned}
&-2a_tPQ_t - a\cdot P\cdot\partial_Q cQ_t - aPQ_{tt} + \mathrm{i}a(Q_t)^2 - c^2(Q)a\mathrm{i} = 0,\\
&a_tPc(Q) - aP\partial_Q cQ_t - aPQ_{tt} + \mathrm{i}a(Q_t)^2 - c^2(Q)\cdot a\cdot \mathrm{i} = 0,\\
&a_tPc(Q) + aP\partial_Q cc(Q) + aP(c(Q))_t + \mathrm{i}ac^2(Q) - c^2(Q)\cdot a\cdot \mathrm{i} = 0,\\
&a_tPc(Q) + aPc'(Q)c(Q) + aP\partial_Q cQ_t = 0,\\
&a_tPc(Q) + aP\partial_Q cc(Q) - aP\partial_Q cc(Q) = 0,\\
&a_tPc(Q) = 0.
\end{aligned}$$

This means that $a_t = 0$, and a should be a constant. This is where the direct substitution breaks down for solving this method. In fact, a can be further calculated and it can be seen that $a = 0$. Looking at the terms with order $\epsilon^{-1}(x-Q)^2$, the equation is:

$$-\frac{1}{2}(c^2(Q))_{QQ}(x-Q)^2(a\cdot\mathrm{i}) = 0.$$

The only constant a that will make this work is when a = 0. Therefore, using direct substitution as what was used for WKB method and Gaussian Beam Method, the equation of a will be a = 0. This is, however, a trivial solution. Recalling the real situation, a represents the amplitude of the seismic wave, which should not be 0. Finding a trivial solution will not be helpful for solving real world problems. In order to find the non-trivial solution, the following lemma is needed.

Lemma 5.1. *For any value $a(y,q,p)$, the following relation is true:*

$$a(y,q,p)(x-Q) \sim -\epsilon\partial z(aZ^{-1}),$$

where the notation $f \sim g$ is to denote that

$$\int\int\int fe^{\frac{\mathrm{i}}{\epsilon}T}dydqdp = \int\int\int ge^{\frac{\mathrm{i}}{\epsilon}T}dydqdp$$

and z is defined by

$$\begin{aligned}\partial_z &= \partial_q - \mathrm{i}\partial_p,\\ Z &= \partial_z(Q+\mathrm{i}P).\end{aligned}$$

Proof. First of all, from the initial condition, since $Q(0,q,p) = q$ and $P(0,q,p) = p$. The following relations can be obtained at time $t = 0$.

$$\begin{aligned}-(\partial_q Q)P + p &= -(\partial_q p) + p = -p + p = 0,\\ (\partial_p Q)P &= (\partial_p q)p = 0 \times p = 0.\end{aligned}$$

Using the Hamiltonian defined earlier, the partial derivatives respect to t can also be calculated. The partial derivatives for $-(\partial_q Q)P + p$ is as the following

$$\begin{aligned}\partial_t(-(\partial_q Q)P + p) &= -\partial_t\partial_q QP - \partial_q Q\partial_t P,\\ &= -\partial_q\partial_t QP - (\partial_q Q)c'(Q)P,\\ &= \partial_q c(Q)P - (\partial_q Q)c'(Q)P,\\ &= \partial_q c(Q)P - \partial_q c(Q)P,\\ &= 0,\end{aligned}$$

where chain rules are used multiple times in this calculations. Similarly, the partial derivative of $(\partial_p Q)P$ can also be calculated. The calculations are:

$$\begin{aligned}\partial_t((\partial_p Q)P) &= \partial_t\partial_p QP + \partial_p Q\partial_t P,\\ &= \partial_p(-c(Q))P + \partial_p Qc'(Q)P,\\ &= -\partial_p c(Q)P + \partial_p c(Q)P,\\ &= 0.\end{aligned}$$

Therefore, for all $t > 0$, the following relations hold:

$$\begin{aligned}-(\partial_q Q)P + p &= 0,\\ (\partial_p Q)P &= 0.\end{aligned}$$

Then by straightforward calculation of $\partial_q T$ and $\partial_p T$ yield the following results.

$$\partial_q T = s_q + P_q(x-Q) - Q_q P - Q_q \mathrm{i}(x-Q) + p - \mathrm{i}(y-q).$$

As from the above calculation, from $s_t = 0$ and the initial condition $s(0,q,p) = 0$, it can be concluded that $s = 0$. Therefore, omitting this term, and also plugging in the equation that $p = \partial_q QP$, the function will become

$$\begin{aligned}\partial_q T &= P_q(x-Q) - Q_q \mathrm{i}(x-Q) - \mathrm{i}(y-q),\\ \partial_q T &= (\partial_q P - \mathrm{i}\partial_q Q)(x-Q) - \mathrm{i}(y-q).\end{aligned}$$

For $\partial_p T$, using $(\partial_p Q)P = 0$, the result will be

$$\begin{aligned}\partial_p T &= \partial_p P(x-Q) - (\partial_p Q)P - \mathrm{i}(\partial_p Q)(x-Q) - (y-q),\\ &= (\partial_p P - \mathrm{i}\partial_q Q)(x-Q) - (y-q).\end{aligned}$$

Thus the two equations imply that $\mathrm{i}\partial_z T = Z(x-Q)$ because

$$\begin{aligned}\mathrm{i}\partial_z T &= \mathrm{i}(\partial_q - \mathrm{i}\partial_p)T,\\ &= \mathrm{i}((\partial_q P - \mathrm{i}\partial_q Q)(x-Q) - \mathrm{i}(y-q)) + (\partial_p P - \mathrm{i}\partial_q Q)(x-Q) - (y-q),\\ &= i\partial_q P(x-Q) + \partial_q Q(x-Q) + (y-q) + (\partial_p P - i\partial_q Q)(x-Q) - (y-q),\\ &= (\partial_q Q - \mathrm{i}\partial_q Q + \partial_p P + \mathrm{i}\partial_q P)(x-Q),\\ &= (\partial_q - \mathrm{i}\partial_p)(Q + \mathrm{i}P)(x-Q),\\ &= \partial_z(Q + \mathrm{i}P)(x-Q),\\ &= Z(x-Q).\end{aligned}$$

If Z, is a multidimensional vector, then additional proof step to show that Z is invertible is needed. However, since in this paper all the analysis are demonstrated in one-dimension, directly above calculation gives that $(x-Q) = iZ^{-1}\partial_z T$. This equality gives that:

$$\begin{aligned}\int_{R^3} a(x-Q)e^{\frac{\mathrm{i}T}{\epsilon}}\,dydpdq &= \epsilon\int_{R^3} a\frac{\mathrm{i}}{\epsilon}Z^{-1}\partial_z T e^{\frac{\mathrm{i}T}{\epsilon}}\,dydpdq,\\ &= \epsilon\int_{R^3} aZ^{-1}\partial_z(e^{\frac{\mathrm{i}T}{\epsilon}})dydpdq,\\ &= -\epsilon\int_{R^3}\partial_z(aZ^{-1})e^{\frac{\mathrm{i}T}{\epsilon}}\,dydpdq,\end{aligned}$$

where the last equality uses integral by parts. Therefore, lemma 5.1 is proven. □

This lemma will be crucial for calculating the Frozen Gaussian Beam Method. With $a(y,q,p)(x-Q) \sim -\epsilon\partial z(aZ^{-1})$, it can then be proven that

$$\begin{aligned}M(x-Q)^2 &\sim (M(x-Q))(x-Q),\\ &\sim -\epsilon\partial_z(M(x-Q)Z^{-1}),\\ &\sim -\epsilon\partial_z(MZ^{-1})(x-Q) + \epsilon MZ^{-1}\partial_z Q,\\ &\sim \epsilon^2\partial_z(\partial_z(MZ^{-1})Z^{-1}) + \epsilon MZ^{-1}\partial_z Q,\end{aligned}$$

where in the last equality simply plugging in $a = \partial_z(MZ^{-1})$.
Now with these two facts proven, the FGB method can be calculated. Again, plugging in u_{xx} and u_{tt} into the wave equation, the form should be:

$$\frac{1}{2\pi\epsilon}\int\int\int\{a_{tt}+\frac{\mathrm{i}}{\epsilon}2a_t[(P_t-\mathrm{i}Q_t)(x-Q)-Pa_t]+\frac{\mathrm{i}}{\epsilon}a[(P_{tt}-\mathrm{i}Q_{tt})(x-Q)-Q_t(P_t-\mathrm{i}Q_t)$$
$$-\partial_t(PQ_t)]-\frac{1}{\epsilon^2}a[(P_t-\mathrm{i}Q_t)(x-Q)-PQ_t]^2\}e^{\frac{iT}{\epsilon}}dydqdp-(c^2(Q)+(c^2)_Q(x-Q)$$
$$+\frac{1}{2}(c^2)_{QQ}(x-Q)^2+\cdots)\int\int\int\{-\frac{1}{\epsilon^2}a[P+\mathrm{i}(x-Q)]^2-\frac{1}{\epsilon}a\}e^{\frac{\mathrm{i}T}{\epsilon}}dydqdp=0,$$

where $c^2(Q)+(c^2)_Q(x-Q)+\frac{1}{2}(c^2)_{QQ}(x-Q)^2+\cdots$ comes from the Taylor's expansion for $c^2(x)$ around point Q. Again, the first step to solve this asymptotically is to separate the equations by the order of ϵ. This time, the grouping is also based on the lemma 5.1 and facts that can be gotten from lemma 5.1. The first equation is that:

$O(\frac{1}{\epsilon^2})$:

$$-a(PQ_t)^2+c^2aP^2=0,$$

where notice that (x-Q) is treated as having order ϵ and $(x-Q)^2$ is treated as having order ϵ^2. Therefore, these are the only terms for order $O(\frac{1}{\epsilon^2})$. Now by the Hamiltonian, the equation is $Q_t = c(Q)$. Plugging this into the equation, the equation becomes $-a(Pc)^2+c^2aP^2=0$, or $0=0$. Therefore, this does not provide any useful information about a. Looking at the next order, the equation is:

$O(\frac{1}{\epsilon})$:

$$\frac{\mathrm{i}}{\epsilon}2a_t(-Pa_t)+\frac{\mathrm{i}}{\epsilon}a[-Q_t(P_t-\mathrm{i}Q_t)-\partial_t(PQ_t)]-\frac{1}{\epsilon^2}a[(P_t-\mathrm{i}Q_t)^2(x-Q)^2-2(P_t-\mathrm{i}Q_t)PQ_t(x-Q)]$$
$$-c^2(Q)\{-\frac{1}{\epsilon^2}a[2P\mathrm{i}(x-Q)-(x-Q)^2]-\frac{1}{\epsilon}a\}-[(c^2)_Q(x-Q)+\frac{1}{2}(c^2)_{QQ}](-\frac{1}{\epsilon^2}aP^2)\sim 0.$$

Notice that for $O(\frac{1}{\epsilon})$, both $(x-Q)$ and $(x-Q)^2$ are presented in the equation because from lemma 5.1 and the consequence of lemma 5.1, $a(x-Q)$ has a term with ϵ and $M(x-Q)^2$ also has a term with ϵ, and cancelling this with $\frac{1}{\epsilon^2}$ will give the terms with order $O(\frac{1}{\epsilon})$. Now plugging in the result of lemma 5.1 and the corresponding consequence into the equation, the equation becomes:

$$2\mathrm{i}a_t(-PQ_t)+\mathrm{i}a[-Q_t(P_t-\mathrm{i}Q_t)-\partial_t(PQ_t)]-a(P_t-\mathrm{i}Q_t)^2\partial_zQZ^{-1}-2\partial_z[aPQ_t(P_t-\mathrm{i}Q_t)Z^{-1}]$$
$$-\partial_z(2\mathrm{i}c^2aPZ^{-1})-c^2a\partial_zQZ^{-1}+c^2a-\partial_z(aP^2(c^2)_QZ^{-1})+\frac{1}{2}aP^2(c^2)_{QQ}\partial_zQZ^{-1}\sim 0.$$

Plugging in the Hamiltonian condition, the equation becomes

$$2ia_t(-PQ_t)+\mathrm{i}a[c(Q)(c(Q)_QP+\mathrm{i}c(Q))+c(Q)c(Q)_QP-Pc(Q)c(Q)_Q]-a(c(Q)_QP+$$
$$\mathrm{i}c(Q))^2\partial_zQZ^{-1}-2\partial_z[-aPc(Q)(c(Q)_QP+\mathrm{i}c(Q))Z^{-1}]-\partial_z(2\mathrm{i}c^2(Q)aPZ^{-1})$$
$$-c^2(Q)a\partial_zQZ^{-1}+c^2(Q)a-\partial_z(2aP^2c(Q)c(Q)_QZ^{-1})+\frac{1}{2}aP^2(2(c(Q)_Q)^2+$$
$$2c(Q)c(Q)_{QQ})\partial_zQZ^{-1}\sim 0.$$

Now notice that $c(Q)c(Q)_QP$ and $-Pc(Q)c(Q)_Q$ can be cancelled out with each other. Also, $-2\partial_z[-aPc(Q)c(Q)_QPZ^{-1}]$ and $-\partial_z[2aP^2c(Q)c(Q)_QZ^{-1}]$ can be cancelled out with each other. Also $-2\partial_z[-aPc^2(Q)\mathrm{i}Z^{-1}]$ and $-\partial_z(2\mathrm{i}c^2(Q)aPZ^{-1})$ can also be cancelled out. Therefore, after simplification, the equation becomes:

$$2\mathrm{i}a_t(Pc) + \mathrm{i}acc_QP - ac^2 - ac_Q^2P^2\partial_zQZ^{-1} - 2a\mathrm{i}cc_QP\partial_zQZ^{-1} + ac^2\partial_zQZ^{-1} - c^2a\partial_zQZ^{-1} + c^2a + aP^2(c_Q)^2\partial_zQZ^{-1} + aP^2cc_{QQ}\partial_zQZ^{-1} \sim 0.$$

Then cancelling out some extra terms, the equation can be further simplified as:

$$\begin{aligned}
&2\mathrm{i}a_t(Pc) + \mathrm{i}acc_QP - 2a\mathrm{i}cc_QP\partial_zQZ^{-1} + aP^2cc_{QQ}\partial_zQZ^{-1} \sim 0,\\
&2\mathrm{i}Pca_t + cc_QP\mathrm{i}a + (-2\mathrm{i}aPcc_Q + aP^2cc_{QQ})\partial_zQZ^{-1} = 0,\\
&2\mathrm{i}Pca_t = (2\mathrm{i}aPcc_Q - aP^2cc_{QQ})\partial_zQZ^{-1} - cc_QP\mathrm{i}a,\\
&a_t = [(c_Q + \frac{\mathrm{i}}{2}Pc_{QQ})\partial_zQZ^{-1} - \frac{1}{2}c_Q]a.
\end{aligned}$$

This equation can actually be further simplified. Looking at the term $\frac{1}{2}c_Q$, the term can be rewritten using the definition of z defined in the paper earlier as:

$$\begin{aligned}
\frac{1}{2}c_Q &= \frac{1}{2}c_QZZ^{-1},\\
&= \frac{1}{2}c_Q(\partial_zQ + \mathrm{i}\partial_zP)Z^{-1}.
\end{aligned}$$

Therefore, the equation becomes

$$\begin{aligned}
-a_t &= [(-c_Q - \frac{\mathrm{i}}{2}Pc_{QQ})\partial_zQZ^{-1} + \frac{1}{2}c_Q(\partial_zQ + \mathrm{i}\partial_zP)Z^{-1}]a,\\
-a_t &= [(-\frac{1}{2}c_Q - \frac{\mathrm{i}}{2}Pc_{QQ})\partial_zQZ^{-1} + \frac{\mathrm{i}}{2}c_Q\partial_zPZ^{-1}]a.
\end{aligned}$$

Now by observing the following relationship:

$$\begin{aligned}
\partial_tZ &= \partial_z(Q_t + \mathrm{i}P_t),\\
&= \partial_z(-c + \mathrm{i}c_QP),\\
&= -c_Q\partial_zQ + \mathrm{i}c_{QQ}\partial_zQP + \mathrm{i}c_Q\partial_zP,
\end{aligned}$$

Thus, multiplying this by $\frac{a}{2}Z^{-1}$ gives us that

$$\frac{a}{2}Z^{-1}\partial_tZ = -\frac{a}{2}c_Q\partial_zQZ^{-1} + \frac{a\mathrm{i}}{2}c_{QQ}\partial_zQPZ^{-1} + \frac{a\mathrm{i}}{2}c_Q\partial_zPZ^{-1}$$

Thus the equation from previous section can be further reduced to:

$$\begin{aligned}
-a_t &= \frac{a}{2}Z^{-1}\partial_tZ - \frac{a\mathrm{i}}{2}Pc_{QQ}\partial_zQZ^{-1} - \frac{a\mathrm{i}}{2}c_{QQ}\partial_zQPZ^{-1}\\
&= \frac{a}{2}Z^{-1}\partial_tZ - a\mathrm{i}c_{QQ}\partial_zQPZ^{-1}\\
&= \frac{a}{2}Z^{-1}\partial_tZ - a\mathrm{i}P\partial_z(c_Q)Z^{-1}
\end{aligned}$$

The last equation is the equation to solve for a. Combining with all the previous analysis, this completes the derivation the asymptotic derivations for Frozen Gaussian Beam Method.

6 conclusion

In this paper, three methods for solving wave equations with high frequency were derived. People need to solve those equations asymptotically because using numerical method to solve them will require many points, and will not provide a solution with a good accuracy because of the high frequency. High-dimensional wave equations usually do not have an analytic solution. As a result, asymptotic solution to approach problem of this kind is very important. Wentzel-Kramers-Brillouin Method is the easiest way to solve this kind of problem as the initial condition is used to approximate the solution. Guessing that the solution has a form close to the initial condition of the problem, the complicated wave equation can be broken into several ordinary differential equations that are easier to solve. The shortcoming of this method is that when the solution curves form caustics, because of the fact that the approximation do not have a width to deal with curves that are very close to each other, the approximation given by this method will be inaccurate. This shortcoming then inspires the second method, Gaussian Beam Method. By adding an imaginary part to the phase function to make it decays exponentially from the center, or essentially adding the 'width' to the approximation, Gaussian Beam Method can overcome the shortcoming that WKB method has. However, since the imaginary part is only designed to quadratic term of Taylor's expansion, when the wave equation solution spreads too wide, the method will also be inaccurate. The last method, Frozen Gaussian Beam Method, can overcome this shortcoming by fix the width of the approximation.
There are still many problems left to explore. For example, for all of the wave equations tested in this paper, the interface function c is a smooth function. However, in real situation, this is usually not the case. Wave can propagate from one media to another media, and the function c does not necessary have to be smooth. This can be a very interesting problem that requires further studies.

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