An Extension of Zeeman's An Introduction to Topology

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Abstract. The classification of surfaces is one of the oldest and most fundamental building blocks of geometric topology. The goal of this paper is to provide an extension of "An Introduction to Topology: The Classification Theorem of Surfaces" by E.C. Zeeman. Zeeman presents the theorem for connected, closed and triangulable surfaces in an intuitively simple and visual, yet thorough manner to students with no topology background. In this paper I will more formally and rigorously present the proper tools needed to prove the Classification Theorem of closed surfaces, which states that any closed surface is homeomorphic to a representative surface. I will approach the theorem and the topics leading up to it with more extensive details, and appropriate references.

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Contents

1. Surfaces

In topology, a surface is a Hausdorff topological space in which every point has a neighborhood homeomorphic to an open subset of the Euclidean plane. All our surfaces will have the following properties:
(i) connected
(iii) compact

Definition 1.1. *A toplogical space is disconnected if it can be represented as a union of disjoint, nonempty open subsets. Otherwise it is a connected topological space.*

Definition 1.2. *Let X be a topological space. A continuous function $f : [0, 1] \to X$ is a path from $f(0)$ to $f(1)$.*

Definition 1.3. *X is path connected if and only if for each $x, y \in X$ there is a path from x to y.*

Theorem 1.4. *Every path connected surface S is connected.*

Proof. Suppose S is a path connected surface but not connected. Then there exists nonempty, disjoint, open subsets $U, V \in S$ such that $X = U \cup V$ Let $x \in U$ and $y \in V$ Since S is path connected there exists a continious function $f : [0, 1] \to S$ such that $f(0) = x$ and $f(1) = y$ Consider $f^{-1}(U)$ and $f^{-1}(V)$ Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are both open in $[0, 1]$ Furthermore, since U and V are disjoint, $f^-1(U)$ and $f^-1(V)$ are also disjoint. And since $X = U \cup V$, $[0, 1] = f^{-1}(U) \cup f^{-1}(V)$ Hence we can write $[0, 1]$ as a union of disjoint, nonempty, open subsets, meaning $[0, 1]$ is disconnected. However this is a contradiction. Therefore every path connected surface must be connected. □

2. Triangulabilty

In Zeeman's paper it was assumed our surfaces are triangulable. We will illustrate the argument in The Jordan-Schönflies Theorem and Classification of Surfaces by Carsten Thomassen that all surfaces can be triangulated. Let's first review what this means.

Definition 2.1. *A simplicial complex, K, is a set of simplices (triangles, or tetrahedrons of arbitrary dimension) that satisfy the following:*
(i) any face of a simplex in K is also in K
(ii) the intersection of two simplices $k_1, k_2 \in K$ is a face of both k_1, k_2

Definition 2.2. *A triangulation of a surface S is a simplicial complex K homeomorphic to S along with a homeomorphism $h : K \to S$. We say K is a triangulated surface.*

Notation: If S is a triangulated surface, we will denote that triangulation by $|S|$.

A natural question is why can any surface be triangulated?

Theorem 2.3. *Every surface S is homeomorphic to a triangulated surface.*

Proof. Let S be a surface, and $p \in S$. Let $D(p)$ be a disk in the Euclidean plane which is homeomorphic to a neighborhood of $p \in S$. To avoid excessive notation we will denote a point in $D(p)$ the same as the corresponding point on S.

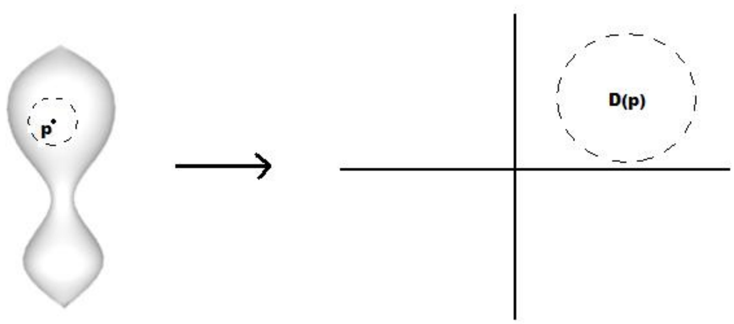

Figure 1: $D(p)$ is homeomorphic to a neighborhood of $p \in S$

In $D(p)$ we will draw two quadrangles (polygon with four sides and four vertices) $Q_1(p)$ and $Q_2(p)$ such that $p \in int(Q_1(p)) \subset int(Q_2(p))$. Here $Q_i(p)$ is the union of the 4 sides of the quadrangle, but does not contain the interior.

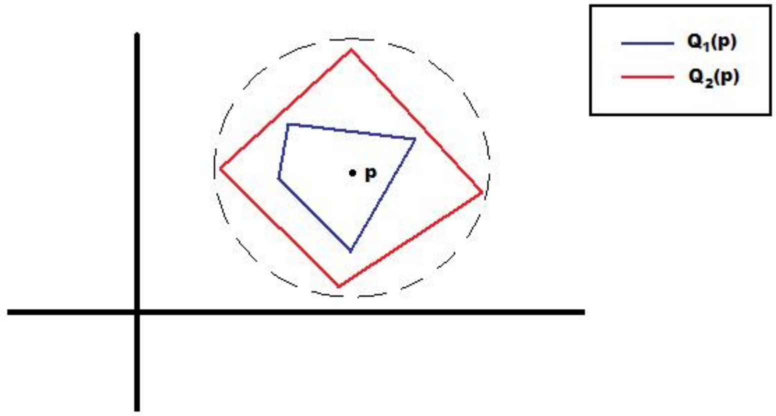

Figure 2: Quadrangles $Q_1(p)$ and $Q_2(p)$

Since S is compact it has a finite number of points $p_1, p_2, ..., p_n$ such that $S = \bigcup_{i=1}^{n} int(Q_1(p_i))$. $D(p_1), D(p_2), ..., D(p_n)$ can be assumed to be pairwise disjoint in the plane. We want to show that $Q_1(p_1), Q_1(p_2), ..., Q_1(p_n)$ can be chosen such that they form a 2-cell embedding of S. By induction suppose $Q_1(p_1), ..., Q_1(p_n)$ have been chosen such that any two of $Q_1(p_1), Q_1(p_2), ..., Q_1(p_{k-1})$ have only a finite number of points in common on S. Now lets focus on $Q_2(p_k)$. Define a bad segment, denoted as b, of $Q_1(p_j)$, $1 \leq j \leq k-1$, which joins 2 points of $Q_2(p_k)$, and the whole segment lies within $int(Q_2(p_k))$.

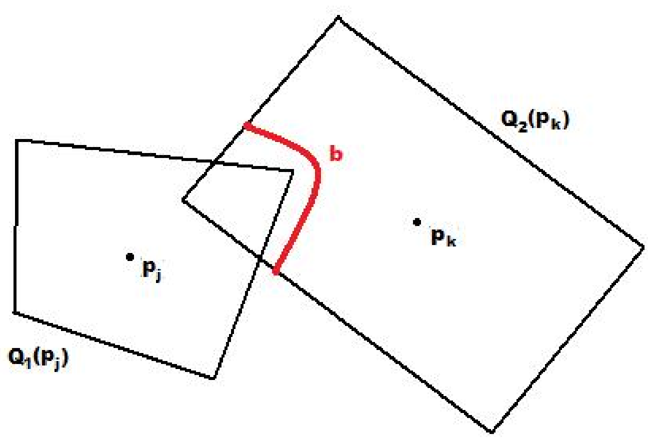

Figure 3: Bad segment b

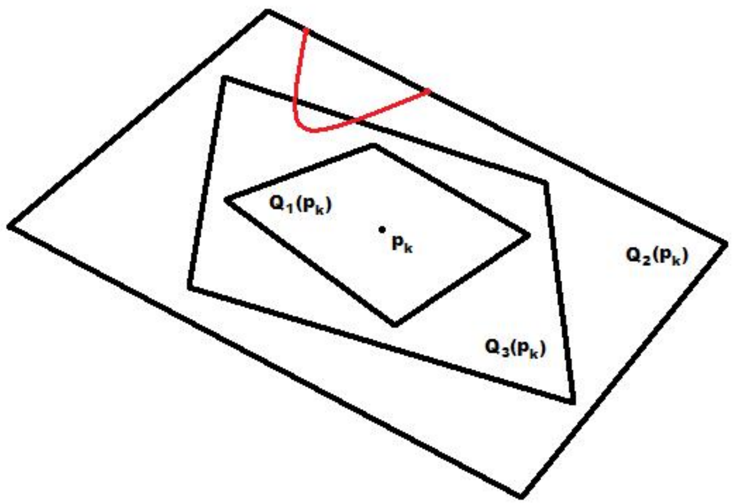

Figure 4: Very bad segment

Let $int(Q_1(p_k)) \subset int(Q_3(p_k)) \subset int(Q_2(p_k))$. A segment in $int(Q_2(p_k))$ is very bad if it intersects $Q_3(p_k)$.
There may be infintely many bad segments, but only finitely many very bad ones. The very bad segments together with $Q_2(p_k)$ form a 2-connected graph, Γ. A graph is 2-connected if there does not exist a vertex whose removal disconnects the graph. Now redraw Γ inside $Q_2(p_k)$ such that we get a graph Γ' which is plane isomorphic to Γ and all edges of Γ' are simple polygonal arcs (finite union of straight arcs).

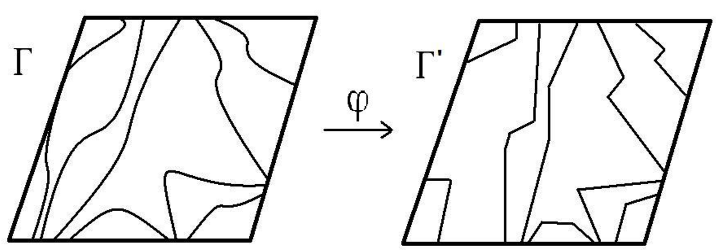

Figure 5: Isomorphism ϕ

By the Jordan Schoenflies Theorem [Th], we can extend the plane isomorphism ϕ to a homeomorphism of $int(Q_2(p_k))$ keeping $Q_2(p_k)$ fixed. This transforms $Q_1(p_k)$ and $Q_3(p_k)$ into simple closed curves (non-self intersecting) Q_1' and Q_3' such that $p_k \in int(Q_1') \subset int(Q_3')$.

Figure 6:

Now consider a simple closed polygonal curve $Q_3'' \subset int(Q_2(p_k))$ such that $Q_1' \subset int(Q_3'')$ and Q_3'' intersects no bad segments except the very bad segments.

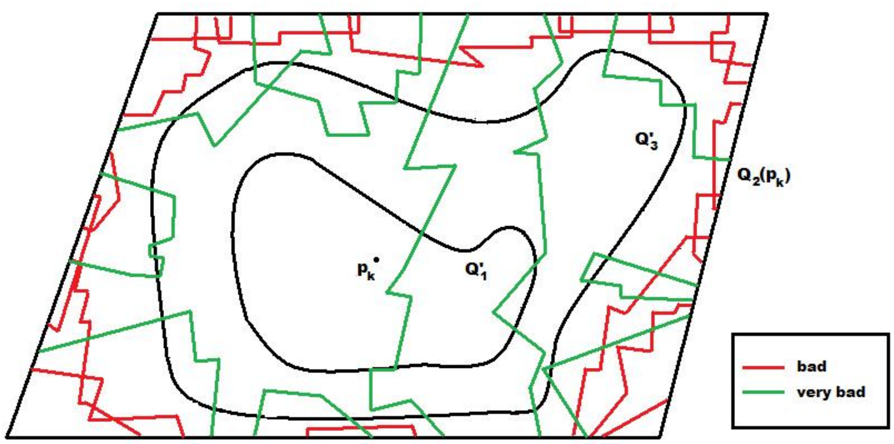

Figure 7:

If we let Q_3'' be the new choice for $Q_1(p_k)$, then any two of $Q_1(p_1)$,...,$Q_1(p_k)$ have only finite intersection. The inductive hypothesis is proved for all k. Thus we can assume there are only finitely many very bad segments inside each $Q_2(p_k)$, and that those segments are polygonal arcs forming a 2-connected plane graph. The union $\bigcup_{i=1}^{n} Q_1(p_i)$ may be thought of as a graph $\Gamma*$ on S. Each region of $S\backslash\Gamma*$ is bounded by a cycle C in Γ.

Now draw a convex polygon C' with side length one so that the corners of C' correspond to vertices of C. The union of C' form a surface S', containing a graph Γ', isomophic to $\Gamma*$. Each C can be triangulated by coning to its center. An isomorphism from $\Gamma*$ to Γ' can be extended to a homeomorphism f from the point set of $\Gamma*$ on S onto the point set of Γ' on S'. Notice the restriction of f to C is a homeomorphism onto C'. Thus f can be extended to a homeomorphism of $i\bar{n}t(C)$ to $i\bar{n}t(C')$. Therefore S is homeomorphic to S'. □

3. Classification

To classify surfaces we need to create a list of standard (or representative) surfaces and show that every surface is homeomorphic to one of the standard ones. Homeomorphism is an equivalence relation on the set of all surfaces, and the equivalence classes represent the different types of surfaces.

Definition 3.1. *Two surfaces, X and Y, are homeomorphic if there is a one-to-one continuous function between them, and a continuous inverse function.*
Notation: $X \cong Y$

Example 3.2. *A sphere X is homeomorphic to an ellipsoid Y by radial projection $X \to Y$.*

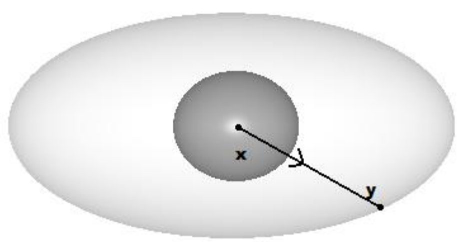

Figure 8: Radial projection of a sphere

Example 3.3. *Constructing a homeomorphism between a knotted surface and an unknotted surface.*
Suppose we make a cut in a surface X, and later sew the cut up again after deformation of X. Then the resulting surface is homeomorphic to X.

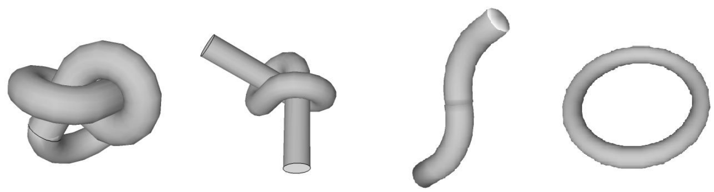

Figure 9: Homeomorphism between a knotted surface and an unknotted surface

Definition 3.4. *The genus of a connected and orientable surface is the maximum number of nonintersecting simple closed curves that can be drawn on the surface without disconnecting it.*

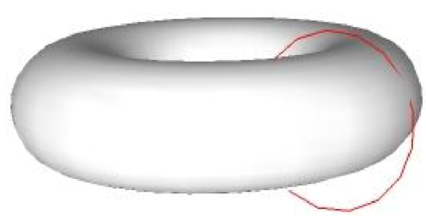

Figure 10: The genus of a torus is 1

Informally, the genus of a surface is equal to the number of holes or handles in the surface.

Example 3.5.

$$\begin{aligned} \textit{genus of sphere} &: 0 \\ \textit{genus of torus} &: 1 \\ \textit{genus of pretzel} &: 2 \end{aligned}$$

Definition 3.6. *The genus of a connected and non-orientable surface is the number of cross-caps (Möbius strips) sewn on to a sphere.*

Example 3.7. *genus 0 is omitted because this is a sphere, which is orientable*
genus 1 is a real projective plane
genus 2 is a Klein bottle

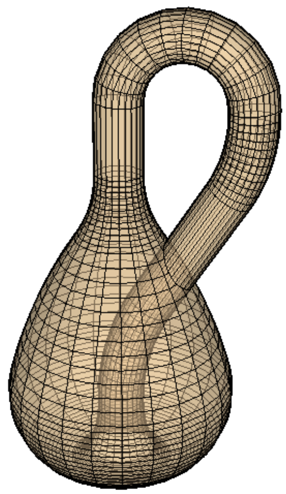

Figure 11: Klein bottle

Theorem 3.8. *Classification Theorem*
Any connected, compact surface is homeomorphic to one of the standard surfaces described above.

Before proving this theorem we need to establish some definitions and lemmas.

Definition 3.9. *Let S be a connected surface and, following the discussion above, choose a triangulation, $|S|$, of S. (By a curve on $|S|$, we mean a closed, non-selfintersecting path consisting of vertices and edges of the triangulation) A curve* ***separates*** *S if cutting along the curve disconnects S into two pieces. S is* ***spherelike*** *if every curve (in every triangulation) separates S.*

Example 3.10. *A sphere is spherelike (Jordan Curve Theorem)*

Example 3.11. *A torus is not spherelike because the following curve does not separate it.*

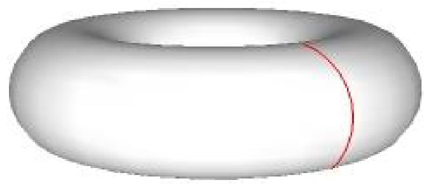

Figure 12: A torus is not spherelike

Definition 3.12. *Let S be a surface and choose a triangulation, $|S|$, of S. Let v be the number of vertices, e be the number of edges, and f be the number of faces of $|S|$. The **Euler characteristic** of $|S|$, $\chi(|S|)$, is defined by*

$$\chi(|S|) = v - e + f$$

Note: It will follow from our argument that $\chi(|S|)$ is independent of choice of triangulation.

4. Graphs

We must introduce the concept of graphs before we can prove any lemmas for the Classification Theorem.

Definition 4.1. *A graph is a finite collection of connected vertices and edges.*

Definition 4.2. *A path in a graph is a sequence of vertices such that from each vertex there is an edge that connects it to the next vertex in the sequence. A simple path is a path with no repeated vertices.*

Definition 4.3. *A tree is a graph in which any two vertices are connected by only one simple path. In other words, it is a graph that contains no cycles.*

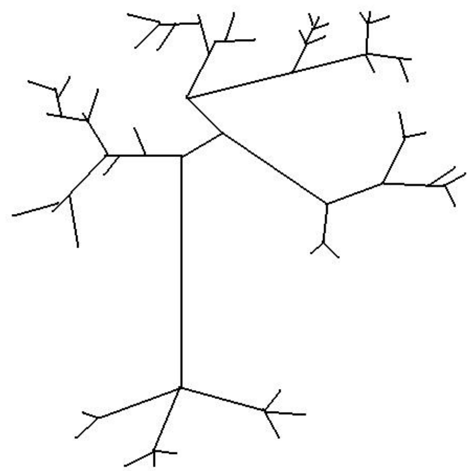

Figure 13: A tree

Note: Every tree contains at least one end vertex (a vertex on only one edge), otherwise any path longer than the number of edges in the tree would contain a cycle.

The Euler characteristic of a graph G with v vertices and e edges is:

$$\chi(G) = v - e$$

Lemma 4.4. *Let T be a tree. Then $\chi(T) = 1$.*

Proof. (Induction) Let $e = 0$. T is just a point, hence $\chi(T) = 1-0 = 1$.
Suppose $\chi(T) = 1$ holds for $e - 1$ edges.
Now let T have e edges. Choose one of its end vertices, and remove it along with the edge attached to it. This will not change $\chi(T)$ since v and e are both reduced by 1. What remains is a new tree, T_1, with $e - 1$ edges.
By the inductive hypothesis, $\chi(T_1) = \chi(T) = 1$. □

Lemma 4.5. *Let G be a graph containing a cycle. Then $\chi(G) < 1$.*

Proof. Since G contains a cycle, we can remove an edge from that cycle without disconnecting G. Call this altered graph G_1.
Thus $\chi(G_1) = v - e + 1 = \chi(G) + 1$. So $\chi(G) = \chi(G_1) - 1$.
Either G_1 contains no more cycles and is hence a tree, or we can remove another edge to form a a new graph G_2. We can remove a finite number of edges til we get a tree. Thus after n steps, we get a tree G_n,$n \geq 1$.

Then

$$\begin{aligned}\chi(G) &= \chi(G_n) - n \\ &= 1 - n \\ &< 1\end{aligned}$$

□

5. Dual Triangulation

Let S be a connected closed triangulable surface, and choose a triangulation of S. Within each triangle, X, choose an interior point x, and call it a dual vertex of X. If two triangles X and Y share an edge, E, join their dual vertices x and y, forming a dual edge xy. xy intersects E once and does not meet any other edges.

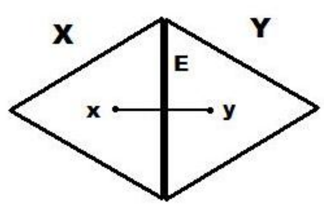

Figure 14:

Definition 5.1. *A tree consisting of dual-vertices and dual edges is called a dual tree.*

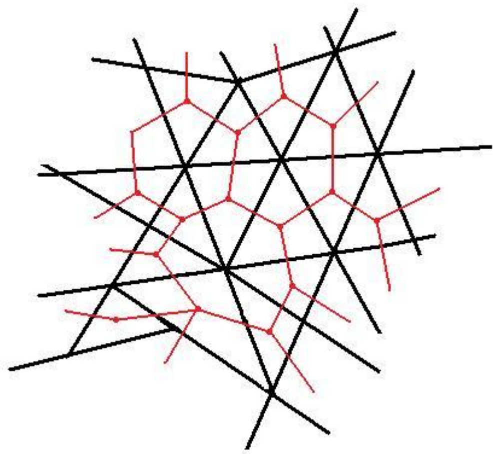

Figure 15: Dual tree

Definition 5.2. *The complement K of a dual tree T is the set of all vertices, edges and triangles of S that do not touch T.*

Lemma 5.3. *K contains all vertices of a triangulation of S.*

Lemma 5.4. *The complement K of a dual tree T is connected.*

Proof. Since the complement K contains all the vertices of S, it is sufficient to show that any two vertices on S can be connected by a path in K. We will use induction on n, the number of edges in T. When $n = 0$, T consists of one dual-vertex. So K has all edges of S. Since S is connected, then K is connected. Assume that this result holds for $n - 1$ edges. Now let T be a dual tree with n edges. Since a tree always contains at least one end vertex, choose one dual-vertex x, and let y be the dual-vertex connected to x. Then xy is the dual-edge of T containing x. Let X, Y be the triangles with dual-vertices x,y respectively, and let the vertices of X be a, b, c.

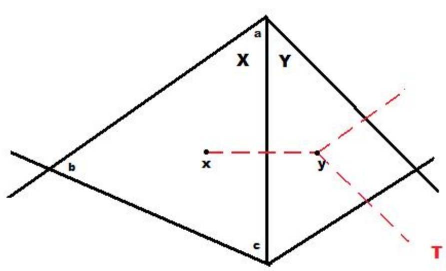

Figure 16:

Let T_1 be the dual tree obtained by removing x and xy from T, and let K_1 be the complement of T_1. Since T_1 has $n-1$ edges, K_1 is connected by the inductive hypothesis. But K is obtained from K_1 by removing triangle X and edge ab. Notice that removing X does not disconnect K_1 because any path in K_1 containing ab can be replaced by a path using ac and cb. Therefore K is connected. □

Definition 5.5. *A maximal dual tree is a dual tree that cannot be made larger without ceasing to be a tree.*

Lemma 5.6. *A maximal dual tree contains all the dual-vertices.*

Proof. Let T be a maximal dual tree and suppose T does not contain the dual-vertex x. Let P be a path from x to any point of T. By shifting P slightly we can ensure it does not go through any vertices of the triangulation.

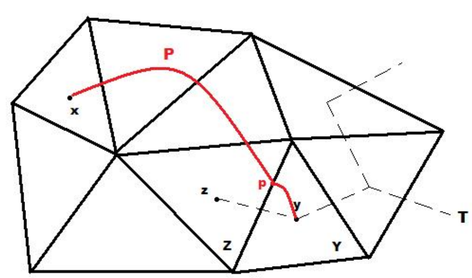

Figure 17:

Let Y be a triangle whose dual-vertex y lies in T. And let p be the point where P first crosses into triangle Y. p must lie on some edge of Y. Let Z be the other triangle containing this edge, with the dual-vertex z. Thus z does not lie in T, otherwise p would not have been the first point. Let T_1 be the dual tree obtained by adding z and yz to T. Hence T_1 is a larger tree than T, and therefore T is not maximal. This leads to a contradiction. □

Lemma 5.7. *Let T be a maximal dual tree and K be its complement. Then K contains no triangles.*

Lemma 5.8. *Let T be a maximal dual tree and K be its complement. Then $\chi(|S|) = \chi(K) + \chi(T)$.*

Proof. By Lemma 5.6, the faces of $|S|$ correspond exactly to the vertices of T, so $f_S = v_T$. By Lemma 5.3, the vertices of $|S|$ correspond exactly to the vertices of K, so $v_S = v_K$. Each edge in S either is crossed by exactly one edge in T, or it lies in K. Hence $e_S = e_K + e_T$. Adding this up gives:

$$\begin{aligned}\chi(|S|) &= f_S - e_S + v_S \\ &= v_T - (e_k + e_T) + v_S \\ &= (v_T - e_T) + (v_K - e_K) \\ &= \chi(T) + \chi(K).\end{aligned}$$

□

6. Classification Theorem

Now return to the Classification Theorem and prove some lemmas.

Lemma 6.1. *If S is a connected, closed, triangulated surface then $\chi(|S|) \leq 2$.*

Proof. Take a triangulation of S with v vertices, e edges, and f faces. Take a maximal dual tree T and it's complement K. We know that K has v vertices and so at least $v - 1$ edges since K is connected. Remember that each dual vertex corresponds to a face of a triangulation of S, so T has f vertices. And by lemma 4.5, T has $f - 1$ edges because it is a tree. e is the sum of the edges of T and K.
So $e \geq v - 1 + f - 1 = v + f - 2$. Thus

$$\begin{aligned}\chi(S) &= v - e + f \\ &\leq v - (v + f - 2) + f \\ &= 2\end{aligned}$$

□

Lemma 6.2. *If S is a connected, closed, triangulated surface then the following are equivalent:*
(i) $|S|$ is spherelike
(ii) $\chi(|S|) = 2$
(iii) S is homeomorphic to a sphere.

Proof. Let's first prove (i) implies (ii). Assume S is spherelike, and $X(|S|) \neq 2$. Let T be a maximal dual tree and K be its complement. Then by Lemma 5.8 $\chi(K) = \chi(|S|) - \chi(T) = \chi(S) - 1 \neq 1$. Thus K is not a tree, and must contain at least one loop, C. C corresponds to a curve on S. Since S is spherelike, this curve disconnects S into pieces. Each triangulation of each piece of S must contain at least one triangle , and thus one dual vertex. By Lemma 5.6 all dual vertices are contained in T, and any two dual vertex can be connected by a path in T since T is a tree. Since the path lies within T, it cannot meet its complement K, and hence C, which lies in K. But this means the pieces of S can be connected by a path that does not touch C. In other words, C does not separate S, which is a contradiction.

Now let's prove (ii) imples (iii). Assume $\chi(S) = 2$. Let T be a maximal dual tree and K be its complement. Since T is a tree, by Lemma 5.8, $\chi(K) = \chi(S) - \chi(T) = 2 - 1 = 1$. So by Lemma 4.5, K is a tree. Let $N(T)$ be a neighborhood of T. We claim $N(T)$ is homeomorphic to a disk. To prove this we start by continuously removing edges of T, reducing it down to a single point. Now we put a disk around the point. By reversing the process we expand the disk with each added edge until we get back T. This shows N(T) is homeomorphic to a disk. Similarly, a neighborhood $N(K)$ is homeomorphic to a disk, since K is a tree. Choosing $N(T)$ and $N(K)$ such that their union is the whole surface and their intersection is the boundary of each results in $S = N(T) \cup N(K)$ being homeomorphic to two disks sewn together, in otherwords, a sphere. This proves (ii) imples (iii).

Finally, to prove (iii) implies (i) we assume S is homeomorphic to a sphere. We want to show any curve C on S separates S. Let C is a polygon consisting of vertices and edges of $|S|$, which we view as finitely many great-circle arcs.

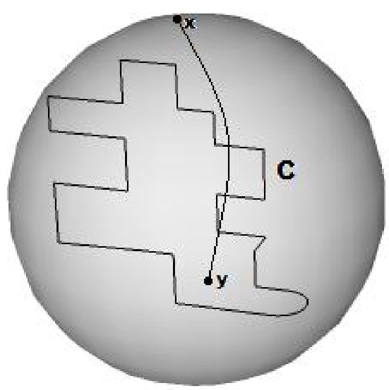

Figure 18:

Let x be a point on S but not on C or any of the great-circle arcs contained in C. We can think of x as the north pole. Define any other point y, not on C nor the south pole, as even or odd according to whether the number of times the arc along S, xy, intersects C is even or odd. By convention, the following kind of intersection counts as 2.

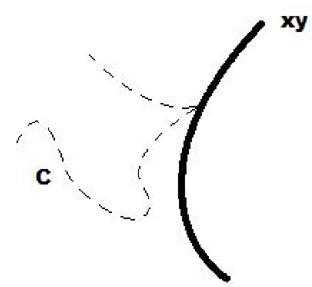

Figure 19:

Along any path not intersecting C, the parity remains constant. So no even point can be joined to an odd point without crossing C. Thus C separates S into two pieces, evens and odds. This completes the proof. □

Now we have all the tools needed to prove the Classification Theorem.

Proof. Let S be a connected, closed, triangulated surface. We need to prove that S is homeomorphic to one of the standard surfaces. Choosing a triangulation of S, we need to compute $\chi(|S|)$. By Lemma 6.1 $\chi(|S|) \leq 2$. If $\chi(|S|) = 2$ then S is homeomorphic to a sphere by Lemma 6.2. So let's assume $\chi(|S|) < 2$. By Lemma 6.2, S is not spherelike, hence we can choose a curve C that will not separate S. Take a small strip of the S containing C. Either the strip is a cylinder or a Mobius strip. If it is a cylinder we will call C an orientation-preserving curve on S, and if it is a Möbius strip we will call C an orientation-reversing curve on S.
We will now construct a new surface S_1. If C is orientation-preserving, cut along C and fill in each side of the cut with a disk. if C is orientation-reversing, cut along C and fill in with one disk, instead of two. The arrows on the boundaries of the disks indicate the direction of the cut. We claim that

$$\chi(|S|) = \begin{cases} \chi(|S|) + 2 & : \text{if C is orientation-preserving} \\ \chi(|S|) + 1 & : \text{if C is orientation-reversing} \end{cases}$$

In order to prove this, assume C has n vertices and n edges. Then $\chi(C) = n - n = 0$. Thus removing C from S does not affect $\chi(|S|)$. First consider the orientation-preserving case. To form S_1, add two disks to each side of the cut. Each disk is formed by joining vertices of C to a single point.

Figure 20: Remove the strip of S containing C and fill each side with a disk

Thus each disk contains $n + 1$ vertices, $2n$ edges, and n triangles. Hence $\chi(\text{disk}) = n+1-2n+n = 1$, and $\chi(|S_1|) = \chi(|S|)+2\chi(\text{disk}) = \chi(|S|) + 2$. In the orientation-reversing case, since only one disk is added, $\chi(|S_1|) = \chi(|S|) + 1$. In either case, $\chi(|S|) < \chi(|S_1|)$.
Either $\chi(|S_1|) = 2$ or $\chi(|S_1|) < 2$. If $\chi(|S_1|) = 2$ then S_1 is homeomorphic to a sphere. If not, we can use a similar procedure as before to produce S_2 such that $\chi(|S_1|) < \chi(|S_2|)$. By Lemma 6.1 the procedure must stop after a finite number of times. So we obtain a sequence of surfaces $S_1, S_2, ..., S_r$ such that $\chi(|S_1|) < \chi(|S_2|) < ... < \chi(|S_r|) = 2$. So S_r contains a number of disjoint disks. Choose a homeomorphism from S_r to the sphere, S_r^*. Now let's focus on the disks in S_r^*. There are three possible situations we can run into:
(i) Two disks with arrows in the opposite directions.
(ii) One disk.
(iii) Two disks with arrows pointing in the same direction.
In the first situation, remove the disks and stretch cylinders from each hole and sew together, creating a handle on the sphere.

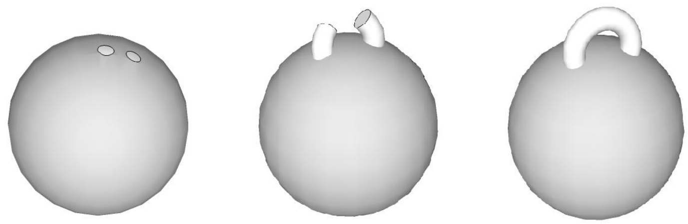

Figure 21: Situation (i)

In situation (ii), remove the disk and sew the boundary of the cut diametrically, sewing a Möbius strip on to the sphere.

In the final situation, remove the two disks, stretch a cylinder out from one hole, and stretch another cylinder out within the sphere from the other hole. Bend the two cylinders til they meet each other at the surface of the sphere, and sew them together. This is essentially sewing on a Klein bottle to the sphere.

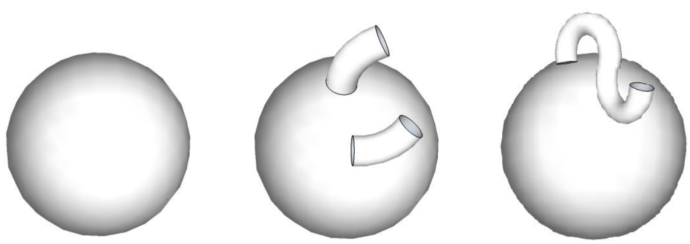

Figure 22: Situation (iii)

Performing all these procedures at once will result in a surface S^* homeomorphic to S.
If S is orientable then so is S^*. So only procedures similar to situation (i) could have occured. Thus S is a standard orientable surface, with genus n, number of surgeries performed or handles sewn on. And
$n = 1 - \dfrac{\chi(|S|)}{2}$
If S is non-orientable then all three of the situations above could have occured, specifically situation (ii) and/or (iii) must have occured. First perform the procedures from (ii) and (iii), then consider situation (i). We can transform situation (i) into (iii) simply by moving one of the disks found in a pair to a Möbius strip that has already been sewn on, move it along the Möbius strip and back to its original position. The arrow along that disk is now heading in the opposite direction, giving

us situation (iii). Thus S^* is a standard non-orientable surface of genus n and $n = 2 - \chi(|S|)$
This completes the proof of the Classification Theorem. $\square$

References

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